Demazure Crystals and Schubert Polynomials

Thesis Presentation

Yuxuan Sun

Haverford College

Spring 2023

Motivation

Schubert
polynomials

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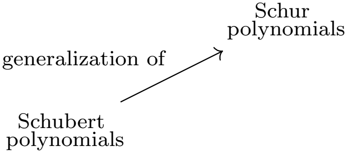
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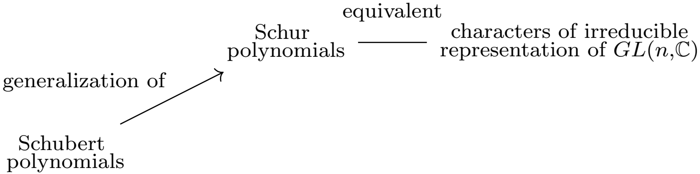
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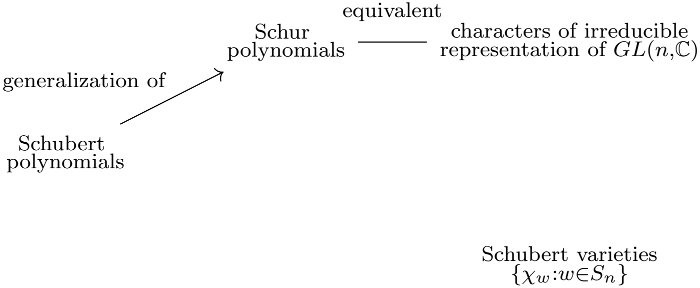
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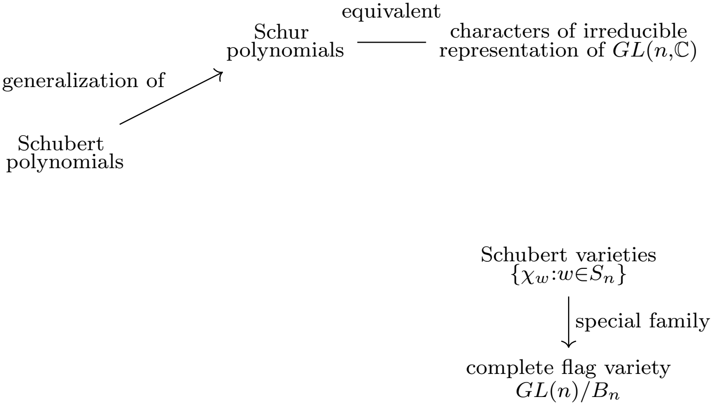
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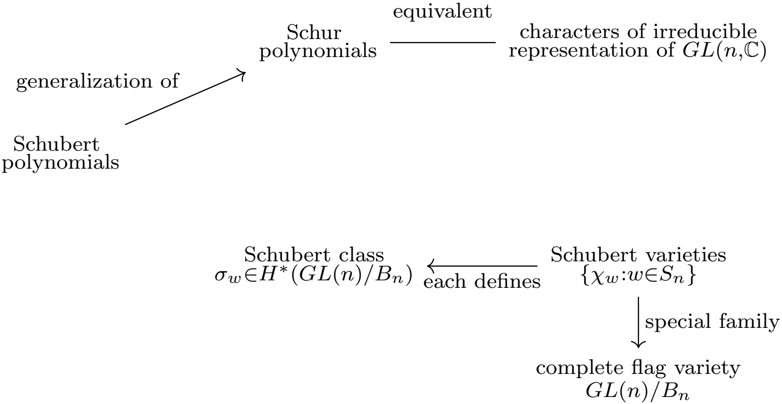

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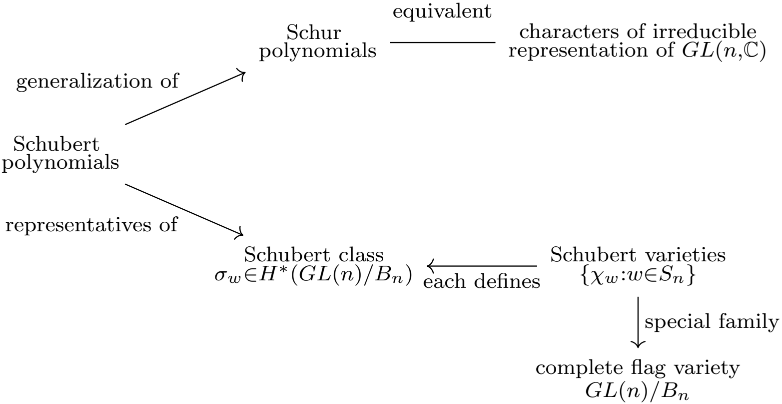

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4. Conjecture about (2) and (3)

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Recall $s_i = (i\ i+1)$

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$$3431 \quad 4341 \quad 3413 \quad 3143 \quad 1343 \quad 4314 \quad 4134 \quad 1434.$$

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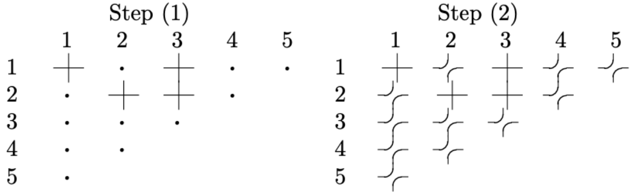

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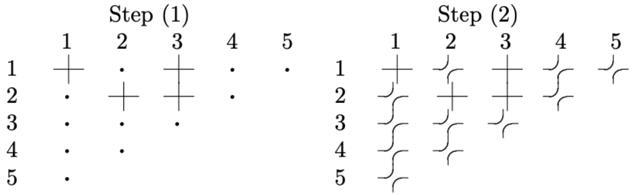

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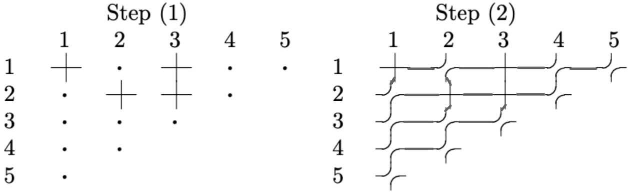

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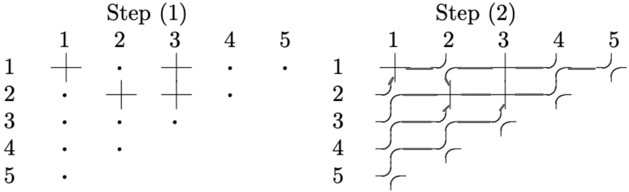

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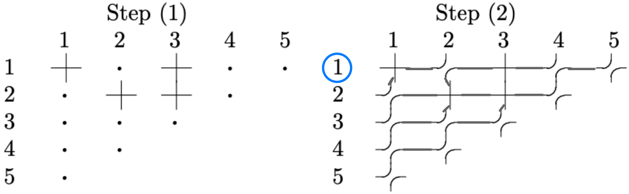

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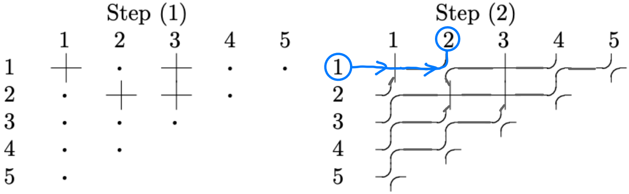

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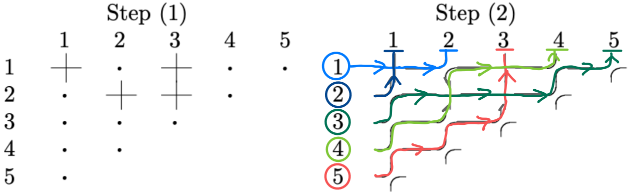

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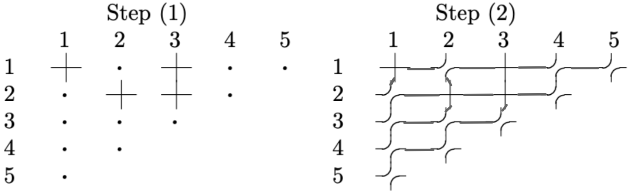

Figure: An element in PD$([2, 1, 5, 4, 3])$

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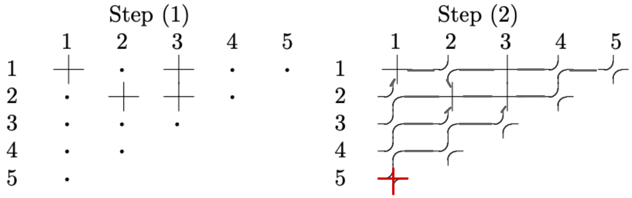

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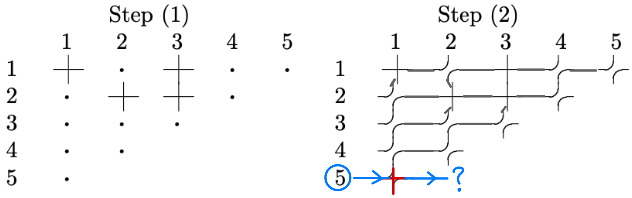

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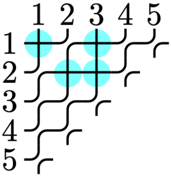

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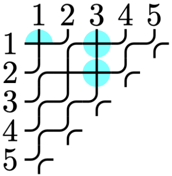

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Theorem (Bergeron and Billey 1993)

$$\mathfrak{S}_w = \sum_{pd \in \mathrm{PD}(w)} x_i^{\mathrm{wt}(pd)_i}.$$

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With conditions:

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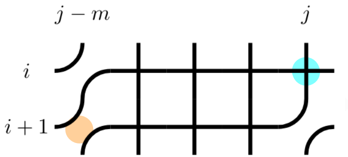

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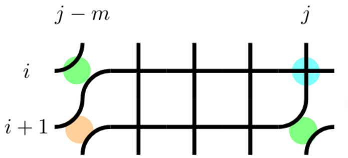

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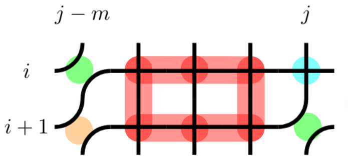

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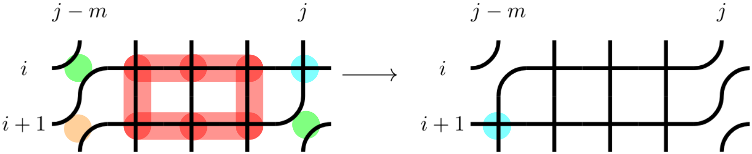

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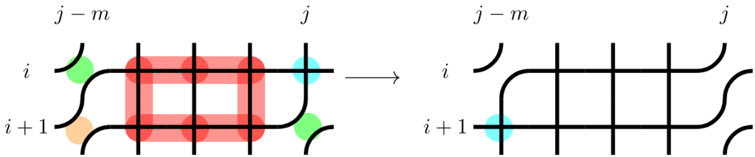

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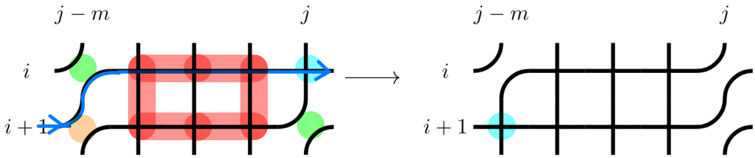

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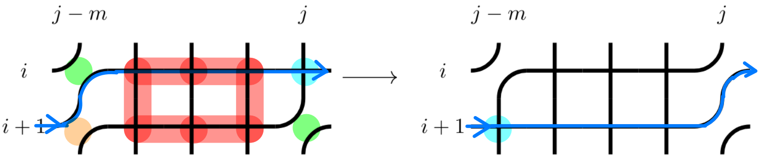

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Recall a chute move C_{ij}^{m}

$$(i, j) \mapsto (i + 1, j - m).$$

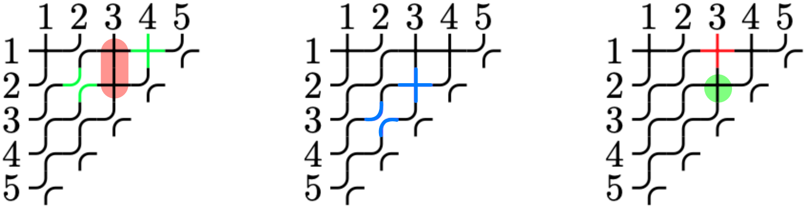


Figure: Two chute moves (C^2_{14}, C^1_{23})and a cross that can't be move

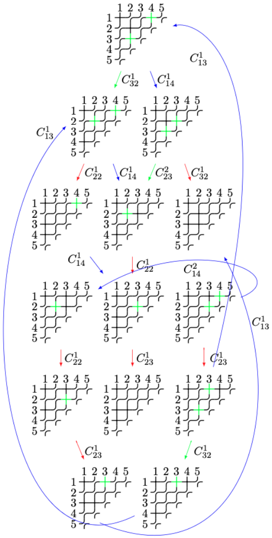
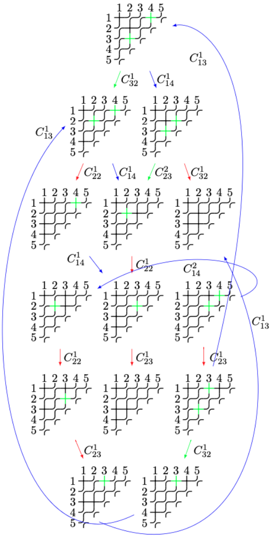


Figure: PD($[2,1,5,4,3]$) with all possible chute moves

Nice facts!



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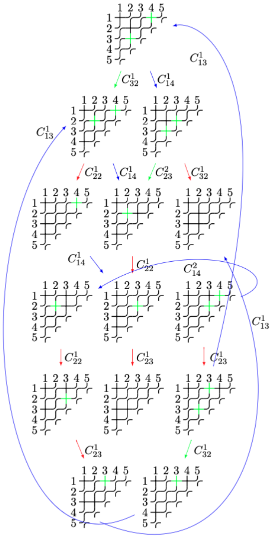

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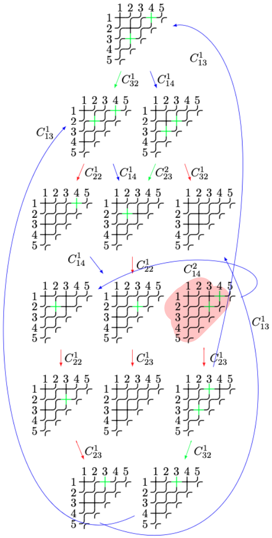

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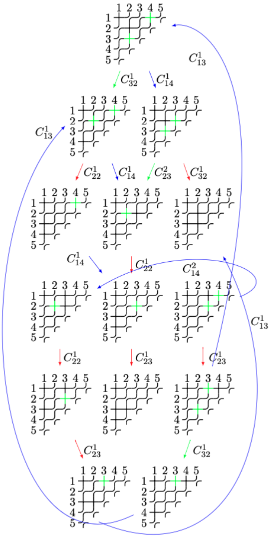

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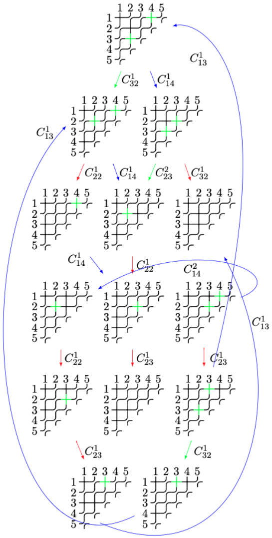

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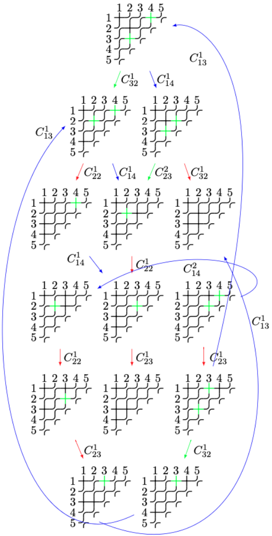

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Chute moves C_{ij}^m are indexed by too many stuffs!

Chute moves have many rules.

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Reduced factorizations with cutoff can do a better job!

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For $[2, 1, 5, 4, 3]$, we partition the word 3431 into 4 blocks.

$|\ |34|3|1|$ is legal $\quad |3|4|3|1|, |\ |34|\ |31|$ are not .

Operators on RFC

A *lowering operators* f_i on RFCs has facts:

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Recall chute move C_{ij}^{m} will move a cross at (i,j) to $(i+1, j-m)$.

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Theorem (S.)

There is a weight-preserving bijection ϕ between $\mathrm{RFC}(w^{-1})$ *and* $\mathrm{PD}(w)$*. i.e.*

$$\mathrm{wt}(rfc) = \mathrm{wt}(\phi(rfc)) \quad \text{or equivalently} \quad \mathrm{wt}(pd) = \mathrm{wt}(\phi^{-1}(pd)),$$

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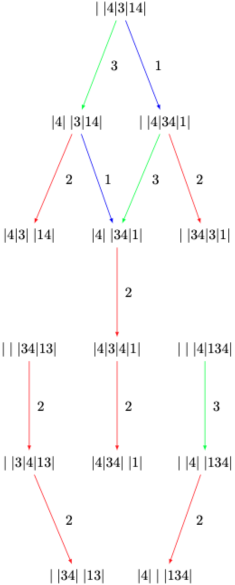
We could study the structure of pipe dreams on RFC!

Crystal on RFC



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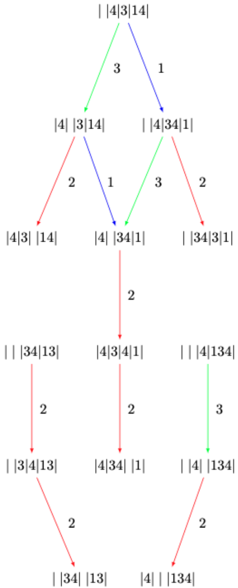

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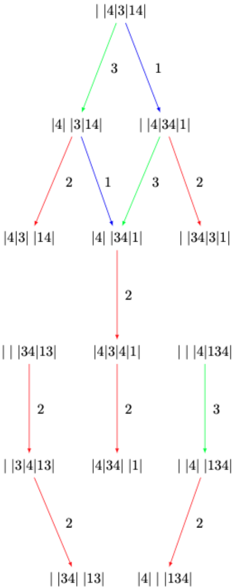

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$$\mathfrak{S}_w = \sum_{pd \in \mathrm{PD}(w)} x_i^{\mathrm{wt}(pd)_i}$$

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$$= \sum_{\alpha} \kappa_\alpha$$

Verify

$$\mathfrak{S}_{[2,1,5,4,3]} = x_1^3x_2 + x_1^2x_2^2 + x_1^3x_3 + 2x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_3^2 + x_1x_2x_3^2 + x_1^3x_4 + x_1^2x_2x_4 + x_1x_2^2x_4 + x_1^2x_3x_4 + x_1x_2x_3x_4 + x_1x_3^2x_4$$

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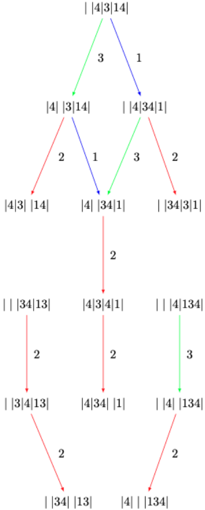
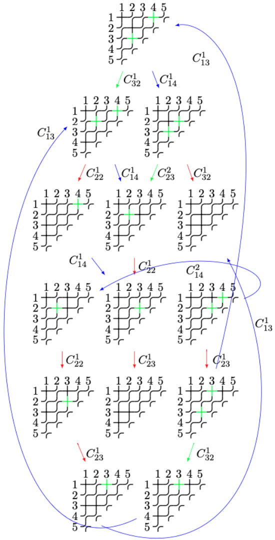
$$\mathfrak{S}_{[2,1,5,4,3]} = \kappa_{(2,0,2,0)} + \kappa_{(3,0,0,1)} + \kappa_{(1,0,2,1)}.$$

Future

Goal: set $C_{ij}^{m} = f_i$





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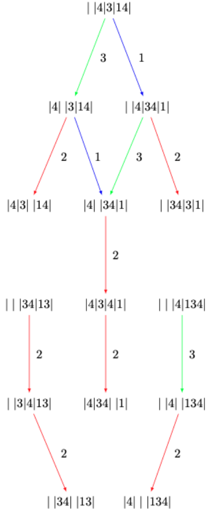

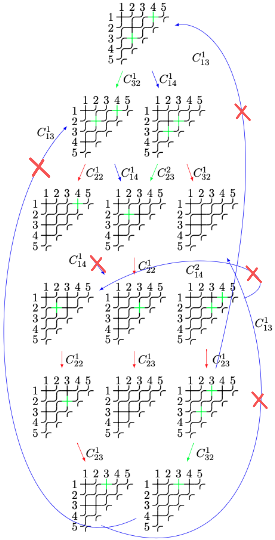

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