Algebra 2 Representationl Theory Notes

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1 Group Representations

Definition 1.1: Representation

A homomorphism

$$\rho : G \to GL_n(\mathbb{C})$$

is called a representation of G of degree/dimension n

Key idea: (group theory) to (linear algebra)

Example 1.2: Sign Representation

$$\rho_s : S_n \to GL_1(\mathbb{R})$$

by

$$\pi \mapsto \begin{cases} [1] & \pi \text{ even} \\ [-1] & \pi \text{ odd} \end{cases}$$

Definition 1.3: permutation representation

Let G be a group which acts faithfully on a set S. The map

$$\begin{aligned} \phi : & G \to Perm(S) \quad \text{by} \\ & g \mapsto \pi_g \end{aligned}$$

Is called the permutation representation of G associated to the action G acts on S.

Definition 1.4: Standard Rpresentation

if G acts on $\mathbb{C}^n$ by linear transformation then G admits the standard representation:

$$\rho_{std} : G \to GL_n(\mathbb{C})$$

by

$$g \mapsto A_g = [g(e_1) \dots g(e_n)]$$

Definition 1.5: regular representation

The permutation representation for the action G actws on G by left multiplication gives rise to a one-to-one homomorphism $\pi : G \to Perm(G)$. If $|G| = n$, the composition:

$$\begin{aligned} \rho_{\text{reg}} \ &: G \xrightarrow{\pi} S_n \xrightarrow{P} GL_n(\mathbb{R}) \quad \text{by} \\ &g \longmapsto \pi_g \longmapsto P_g \end{aligned}$$

where P_g is the premutation matrix associated to the permutation π_g is then also a one-to-one homomorphism ρ_{reg} called the (left) regular representation of G.

Corollary 1.6: Representation to row operation is homomorphism

$$\rho : G \to GL_n(\mathbb{C})$$

is a homomorphism.

Definition 1.7: G-invariant

Let G acts on $\mathbb{C}^n$. A subspace $V \subset \mathbb{C}^n$ is G-invariant if $G(V) \subset V$, meaning $g * v \in V$ for all $g \in G, v \in V$

Definition 1.8: Irreducible representation

A representation of G is called irreducible if it has no proper G-invariant subspaces.

Definition 1.9: isomorphic representations

Two representations are isomorphic if they differ by a change of basis of $\mathbb{C}^n$

2 characters

Theorem 2.1: characters + isomorphism

G is finite

$$\begin{aligned} \text{reps are isomorphic} &\iff \text{they have the same characters} \\ \text{number of distinct irreducible reps} &\iff \text{number of conjugacy classes in } G \end{aligned}$$

Theorem 2.2: 5 theorems

Denote ρ_i as distinct representation of a finite group G, with characters χ_i

1. irreducible characters are orthonormal,

$$\langle \chi_i, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \chi(j(g) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Example 2.3: A_4 conjugacy classes

(1)
(12)(34), (13)(24), (14)(23)
(123),(124),(134),(234)
(132),(142),(143),(243)

3 Roots of unity

Example 3.1

$$\zeta = \cos\theta + i \sin\theta \quad \text{and} \quad \zeta^3 = 1$$

Definition 3.2: root of unity

The n^{th} roots of unity are the n distinct $z \in \mathbb{C}$ s.t.

$$z^n = 1$$

4 Maschke

Theorem 4.1: Maschke's Thm

Every representation:

$$\rho : G \to GL_n(\mathbb{C})$$

satisfies

$$\rho \cong \rho_1 \oplus \ldots \oplus \rho_{m<\infty}$$

where ρ_i are irreducible representations of G

Example 4.2: what does $\oplus$ mean?

Given:

$$\rho_1 : G \to GL_n(\mathbb{C})$$
$$\rho_2 : G \to GL_m(\mathbb{C})$$

We could define:

$$\rho = \rho_1 \oplus \rho_2 : G \to GL_{n+m}(\mathbb{C})$$

by

$$\rho_g = \begin{array}{c|c} \rho_1(g) & \\ \hline & \rho_2(g) \end{array}$$

Definition 4.3: decomposable

Let

$$\rho : G \to GL_n(\mathbb{C})$$

if

$$\mathbb{C}^n = V_1 \oplus V_2$$

for two G-invariant subspaces, V_1, V_2, then

$$\rho \cong \rho_1 \oplus \rho_2$$

where

$$\rho_i = \rho|_{v_i} \quad \text{a restriction to smaller domain}$$

We say ρ is decomposable