Algebra 2 Representationl Theory Notes

Yuxuan Sun

Spring 2022

Contents

1	Group Representations	2
2	characters	3
3	Roots of unity	4
4	Maschke	4

1 Group Representations

Definition 1.1: Representation

A homomorphism

 $\rho: G \to GL_n(\mathbb{C})$

is called a representation of G of degree/dimension \boldsymbol{n}

Key idea: (group theory) to (linear algebra)

Example 1.2: Sign Representation

 $\rho_s: S_n \to GL_1(\mathbb{R})$

by

$$\pi \mapsto \begin{cases} [1] & \pi \text{ even} \\ [-1] & \pi \text{ odd} \end{cases}$$

Definition 1.3: permutation representation

Let G be a group which acts faithfully on a set S. The map

$$\phi: G \to Perm(S)$$
 by
 $g \mapsto \pi_g$

Is called the permutation representation of ${\cal G}$ associated to the action ${\cal G}$ acts on S.

Definition 1.4: Standard Rpresentation

if G acts on \mathbb{C}^n by linear transformation then G admits the standard representation:

 $\rho_{std}: G \to GL_n(\mathbb{C})$

by

 $g \mapsto A_g = [g(e_1) \dots g(e_n)]$

Definition 1.5: regular representation

The permutation representation for the action G actws on G by left multiplication gives rise to a one-to-one homomorphism $\pi : G \to Perm(G)$. If |G| = n, the composition:

$$\rho_{\text{reg}} : G \xrightarrow{\pi} S_n \xrightarrow{P} GL_n(\mathbb{R}) \quad \text{by} \\
g \longmapsto \pi_g \longmapsto P_q$$

where P_g is the premutation matrix associated to the permutation π_g is then also a one-to-one homomorphism ρ_{reg} called the (left) regular representation of G.

Corollary 1.6: Representation to row operation is homomorphism

$$\rho: G \to GL_n(\mathbb{C})$$

is a homomorphism.

Definition 1.7: G-invariant

Let G acts on \mathbb{C}^n . A subspace $V \subset \mathbb{C}^n$ is G-invariant if $G(V) \subset V$, meaning $g * v \in V$ for all $g \in G, v \in V$

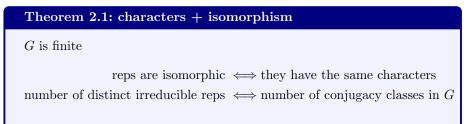
Definition 1.8: Irreducible representation

A representation of G is called irreducible if it has no proper G-invariant subspaces.

Definition 1.9: isomorphic representations

Two representations are isomorphic if they differ by a change of basis of \mathbb{C}^n

2 characters



Theorem 2.2: 5 theorems

Denote ρ_i as distinct representation of a finite group G, with characters χ_i

1. irreducible characters are orthonormal,

$$\langle \chi_i, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \chi(j(g)) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Example 2.3: A_4 conjugacy classes

 $\begin{array}{l} (1) \\ (12)(34), (13)(24), (14)(23) \\ (123), (124), (134), (234) \\ (132), (142), (143), (243) \end{array}$

3 Roots of unity

Example 3.1

 $\zeta = \cos \theta + i \sin \theta$ and $\zeta^3 = 1$

Definition 3.2: root of unity

The n^{th} roots of unity are the n distinct $z \in \mathbb{C}$ s.t.

 $z^n = 1$

4 Maschke

Theorem 4.1: Maschke's Thm
Every representation:
$\rho: G \to GL_n(\mathbb{C})$
satisfies
$ \rho \cong ho_1 \oplus \ldots \oplus ho_{m < \infty} $
where ρ_i are irreducible representations of G

Example 4.2: what does \oplus mean?

Given:

 $\rho_1: G \to GL_n(\mathbb{C})$ $\rho_2: G \to GL_m(\mathbb{C})$

We could define:

$$\rho = \rho_1 \oplus \rho_2 : G \to GL_{n+m}(\mathbb{C})$$

by

$$\rho_g = \frac{\rho_1(g)}{\rho_2(g)}$$

Definition 4.3: decomposable

Let

if

 $\mathbb{C}^n = V_1 \oplus V_2$

 $\rho: G \to GL_n(\mathbb{C})$

for two G-invariant subspaces, V_1, V_2 , then

$$\rho \cong \rho_1 \oplus \rho_2$$

where

 $\rho_i = \rho|_{v_i} ~~$ a restriction to smaller domain

We say ρ is decomposable