Lie Theory

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Contents

1 Background

1.1 Topology

Definition 1.1.1. A topological space is *locally connected* at point x if everyneighborhood of x contains a connected open neighborhood.

2 Topologcial Groups

2.1 Introduction

Definition 2.1.1. A *topological group* is a group such that

1. he product $p : G \times G \to G, p(g, h) = gh$, is a continuous map if $G \times G$ has the product topology;
2. The map $\iota : G \to G, \iota(g) = g^{-1}$, is continuous (hence, a homeomorphism, as $\iota^{-1} = \iota$).

Each element $g \in G$ defines the following maps.

- *left translation*: $L_g : G \to G, L_g(h) = gh$;
- *right translation*: $R_g : G \to G, R_g(h) = hg$;
- *conjugation*: $C_g : G \to G, C_g(h) = ghg^{-1}$.

2.2 Neighborhoods of Identity

An (open) neighborhood of $x \in X$, where X is a topological space, is an open set U that coitains x.

Let G be a topological group, and $1 \in G$ is the identity. $V(1)$ refers to the set of all neighborhoods of 1.

Proposition 2.2.1 (Proposition 2.2)**.** *Let G be a t.g. (topological group), $V = V(1)$. Then we'll have*

1. *(T1)for all $u \in V, 1 \in u$;*
2. *(T2)$u, v \in V \implies u \cap v \in V$;*
3. *(TG1) for all $u \in V$, there exists $v \in V$ s.t. $v^2 \subseteq u$;*
4. *(TG2) $u \in V \implies u^{-1} \in V$;*
5. *(TG3) $u \in V, g \in G \implies gug^{-1} \in V$.*

Definition 2.2.2. Let G be a group, not necessarily topological group. A system of neighborhood of $1 \in G$ is a family of sets sastisfying (T1) to (TG3).

Definition 2.2.3. Let X be a topological space and $x \in X$. A fundamental system of neighborhoods of x is a family F of open sets containing x s.t. for all open u that contains x, there exists $v \in F$ s.t. $v \subseteq u$.

Theorem 2.2.4 (Proposition 2.5)**.** *Let G be an abstract group, V be a system of neighborhoods of 1. There exists a unique topology on G making G into a topological group and s.t. V is a fundamental system of neighborhoods of 1.*

idea of proof. □

Proposition 2.2.5. *Let G be a topological group. TFAE*

1. *topology of G is a Hausdorff*
2. *$\{1\}$ is closed in*
3. $\bigcap_{U \in \mathcal{V}(1)} U = \{1\}$

2.3 Metrizable Groups

Definition 2.3.1. Let G be a topological group. G is metrizable if it has a left-(or right-) invariatn distance which defines the tooplogy left-invariant for all $g \in G$ and $d(gx, gy) = d(x, y)$ for all $x, y \in G$.

Theorem 2.3.2. *A topological grouop G is metrizable iff it has a countable system of neighborhoods of 1.*

2.4 Homomorphisms

We need to talk about $G \to H$ continuous homomorphisms.

Example 2.4.1. *The determinant homomorphism* $\det : GL_n(\mathbb{R}) \to \mathbb{R}^* = GL(1, \mathbb{R})$ *is continuous.*

Theorem 2.4.2. *Let G, H be topological group. A group homomorphism $\phi : G \to H$ is continuous iff ϕ is continuous at $1 \in G$.*

Proof. $\Longrightarrow$ is obvious. Let's look at the other direction.

Note that $\phi \circ L_g = L_{\phi(g)} \circ \phi$ as maps $G \to H$ because

$$(\phi \circ L_g)(g') = \phi(gg') = \phi(g)\phi(g') = (L_{\phi(g)} \circ \phi)(g').$$

Then

$$\phi = L_{\phi(g)} \circ \phi \circ L_{g^{-1}}$$

is continuous at g, as $L_{g^{-1}}$ is continuous at g, ϕ continuous at 1, and $L_{\phi(g)}$ continuous everywhere. □

Theorem 2.4.3. *A map $\phi : G \to H$ is a group homomorphism (G, H are just grouops) iff*

$$gr(\phi) := \{(g, \phi(g)) \mid g \in G\} \subseteq G \times H.$$

Proposition 2.4.4. *Let X and Y be topological spaces, such that Y is Hausdorff. A map $\phi : X \to Y$ is continuous if and only if its graph $gr(\phi)$ is closed and the projection $p(x, \phi(x)) = x$ is a homeomorphism.*

Proof. Suppose ϕ is continuous. Then

$$gr(\phi) = \theta^{-1}(\Delta y) \text{ w.r.t. } \theta : X \times Y \to Y \times Y$$

is closed, since *tehta* is continuous and Δy is closed. □

Theorem 2.4.5. *Suppose G, H are topological groups, H is Hausdorff. The map $G \to H$ is a continuous homomorphism iff $gr(\phi)$ is a closed subgroup and $p : gr(\phi) \to G$ is a homeomorphism.*

2.5 Subgroups

Let G be a topological group. $H \subseteq G$ is a *topological subgroup* if H is a topological group w.r.t. the induced topology.

Proposition 2.5.1. *Let G be a topological group. If $H \subseteq G$ a subgroup, which is open. Then H is also closed.*

Proof. Consider

$$Y = \bigcup_{g \in G - H} gH.$$

Y is open, as it is a union of open sets. H is also closed, as $G - Y = H$. Hence, H is closed. □

Proposition 2.5.2. *G a topological group, $H \subseteq G$ a subgroup. Then $\overline{H}$ is also a subgroup of G.*

Proof. Note that $A \subseteq X$ (subset of a topological space), $x \in \overline{A}$ iff for all open U that contains x, $U \cap A \neq \emptyset$. Then we check the followings.

1. $\overline{H}$ is closed under $m : G \times G \to G$.

□

2.6 Connected Components of Topological Groups

A *connected space* cannot be written as the union of two disjoint open sets.

A *connected component* of a point $x \in X$ is the union of all connected sets containing x, which is also the maximal connected set containing x.

A *connected component* of X is a maximal connected subset.

If $A \subseteq X$ is connected, then the closure $\overline{A}$ is connected. Thus, every connected component is closed.

Let G be a topological group, G_0 is the connected component of $1 \in G$.

Proposition 2.6.1. *G_0 is a closed normal subgroup of G. The connected components of G are exactly gG_0 for $g \in G$.*

A *neighborhood* N of $x \in X$ is a subset $N \subseteq X$, $x \in N$ and there exists an open $U \subseteq X$ s.t $x \in U \subseteq N$.

A space is *locally connected* if for every open neighborhood of every point contains a connected open neighborhood.

Proposition 2.6.2. *If G is locally conencted, then G_0 is open.*

Proposition 2.6.3. *If G connected, $U \in \mathcal{V}(1)$, then $G = \cup_{n \geq 1} U^n$.*

2.7 Group Action

Suppose G a group, X a set.

Definition 2.7.1. A *left action* of a group G on a set X is a function that associates to $g \in G$ a map $a(g) : X \to X$ which satisfies the properties: 1. $a(1) = \mathrm{id}_X$, that is, $a(1)(x) = x$, for every $x \in X$; 2. $a(gh) = a(g) \circ a(h)$.

Definition 2.7.2. Let $\phi_x : G \times X \to X, \phi_y : G \times Y \to Y$. A map $f : X \to Y$ is *G-equivariant* if

$$\phi_y(g, f(x)) = f(\phi_x(g, x)).$$

Same stroy for topological groups.

Definition 2.7.3. Let G be a topological group, X a topological space, an *action* G on X should be continuous. In other words, G acts on X by homeomorphisms ϕ_g.

Action is *transitive* if $X = Gx$ for some $x \in X$. We define the *orbit* of x to be $Gx = \{gx \mid g \in G\}$. A *stabilizer* or *isotropy subgroup* of x is $G_x = \{g \in G \mid gx = x\}$.

An action is an *effective action* or *faithful* if $gx = x, \forall x \in X \implies g = 1$, equivalently, $\cap_{x \in X} G_x = \{1\}$.

Proposition 2.7.4.

$$G/G_x \to X \quad \textit{where} \quad gG_x \mapsto gx.$$

This map is equivariant.

Proposition 2.7.5. *Suppose that the action of G on X is continuous and that X is a Hausdorff space. Then, any isotropy subgroup $G_x, x \in X$, is closed.*

2.8 Homogeneous Spaces

Let G be a topological group.

Definition 2.8.1. A *homogeneous G-space* is just G/H for a subgroup H of G.

Definition 2.8.2. A topological space X without regards to group is *homogeneous* if for all $x, y \in X$, there exists a homeomorphism $\phi : X \to X$ s.t. $\phi(x) = y$.

Topology on G/H is that of a quotient: $\pi : G \to G/H$. In other words, $U \subseteq G/H$ open if $\pi^{-1}(U) \subseteq G$ open.

Note: action of G on G/H is continuous:

$$G \times G/H \to G/H \quad \text{where} \quad (x, gH) \mapsto xgH.$$

Proposition 2.8.3. *We have the following facts.*

1. *G/H is a homogeneous space in the sense of topology.*
2. *$\pi : G \to G/H$ is an open map (it takes open sets to open sets).*
3. *H compact implies that π is a closed map.*
4. *G/H is Hausdorff iff H is closed.*
5. *G/H discrete iff H open. (HW2)*
6. *If G is compact, G/H discrete and finite iff H is open.*
7. *$H \triangleleft G$ implies G/H is a topological group.*
8. *$H := \overline{\{1\}}$. THen H is a normal subgroup of G, and G/H is Hausdorff topological group.*

Proof of 1. Consider left translation

$$L_x : gH \mapsto xgH.$$

This is a homeomorphism since $L_{x^{-1}}$ is an inverse and both are continuous. □

Proof of 2. We need to show that $\pi^{-1}\pi(U)$ is open. (Omitted, just do image preimage and write it as union of right cosets). □

Proof of 3. Take $F \subseteq G$ closed, if H is a compact subset, then $FH \subseteq G$ is closed. (From a proposition from textbook).

Notice that $\pi(F)$ closed iff $\pi^{-1}\pi(F)$ closed, and the latter equals to FH. □

Proof of 4. We first show $\implies$. Note that $H = \pi^{-1}(H)$, which is a point of G/H, so it's closed. Thus H is closed.

Then we show $\Leftarrow$. Consider the homeomorphism

$$f : G/H \times G/H \to G \times G/H \times H \quad \text{where} \quad (g_1H, g_2H) \mapsto (g_1, g_2)H \times H.$$

Denote $\Delta = \{(gH, gH)\}$. Then $f(\Delta) = \{(g, g)H \times H\}$ is closed iff $\pi^{-1}_{G\times G} f(\Delta)$ is closed, which equals to $\{(g_1, g_2) \mid g_1H = g_2H\} = \{(g_1, g_2) \mid g_1^{-1}g_2 \in H\}$. □

Let G be a topological group, $H \subseteq G$ a subgroup.

Proposition 2.8.4. *If H and G/H are compact, then so is G.*

Proof.
$$\pi : G \to G/H$$
is a *perfect map*, i.e., a continuous subjective closed map with compact fibers $\pi^{-1}(x), \forall x \in G/H$. $\square$

Proposition 2.8.5. *If G/H and H are connected, then so is G.*

Proof. Suppose G is not connected, then there exists $A \bigsqcup B = G$, $A, B \neq \emptyset$ open, disjoin $\subseteq G$. Then $\pi(A), \pi(B) \neq \emptyset$, open because π is always open, $\pi(A) \cup \pi(B) = G/H$, which is connected. Therefore $\pi(A) \cap \pi(B) \neq \emptyset$. Thus there exsits $gH \in G/H$ s.t. $gH \cap A \neq \emptyset$ and $gH \cap B \neq \emptyset$. $\square$

2.9 Orbits and Homogeneous Spaces

Homogeneous space G/G_x, we hav ea bijection:
$$G/G_x \to G \cdot x \quad \text{where} \quad gG_x \mapsto gx.$$

Proposition 2.9.1. *Let $G \times X \to X$ be a continuous and transitive action of G on X. Fix $x \in X$ and consider the bijection*
$$\xi_x : G/G_x \to X \text{ given by } \xi_x(gG_X) = gx.$$
Then ξ_x is continuous with respect to the quotient topology in G/G_x.

Proposition 2.9.2. *Let $G \times X \to X$ be a topological transitive group action. Suppose G is locally compact and spearable (i.e., has a countable dense subset) and X is Hausdorff and locally compact, Then*
$$\xi_x : G/G_x \to X = G \cdot x \quad \forall x \in X$$
is a homeomorphism.

2.10 Examples

We have
$$O(N) = \{g \in GL(n, \mathbb{R}) \mid gg^T = I_n(\det g = 1)\}.$$

$O(n)$ acts on $\mathbb{R}^n$ with orbits being $S_r^{n-1} - \{x \in \mathbb{R}^n \mid |x| = r\}, r \geq 0$.

Induction implies that $O(n), SO(n)$ are compact, $SO(n)$ connected.

Also $SL(n, \mathbb{R})$ is connected, as it has for $n > -2$ has 2 orbits on $\mathbb{R}^n : \{0\}, \mathbb{R}^n - \{0\}$. Also $SL(n, \mathbb{C})$ is connected.

Consider unitary groups
$$U(n) = \{g \in GL(n, \mathbb{C}) \mid gg^{-T} - I_n(\det g = 1)\}.$$

$GL(n, \mathbb{F})$ acts on $\mathbb{P}^{n-1}$, which is the set of lines through 0 in $\mathbb{F}^n$.

$Gr_k(n, \mathbb{F})$ is the set of k-dimensional subspaces of $\mathbb{F}^n$, which is the quotient of the set of $n \times k$-matrices of rank k by $GL(k, \mathbb{F})$ acting on the right.

3 Lie Group

3.1 Basics

Definition 3.1.1. A *Lie group* G is a group and a manifold such that

$$m : G \times G \to G$$

is smooth.

The composition of two smooth maps is smooth.

Proposition 3.1.2. *The inverse map $\iota : G \to G$ is a diffeomorphism with*

$$d\iota_g = -(dL_{g^{-1}})_1 \circ (dR_{g^{-1}})_g.$$

Particularly, $\iota_1 = -\operatorname{id}$.

3.2 Tangent Bundle to a Manifold

A fiber bundle is a structure (E, B, π, F), where E, B, and F are topological spaces and $\pi : E \to B$ is a continuous surjection satisfying a local triviality condition outlined below. The space B is called the base space of the bundle, E the total space, and F the fiber. The map π is called the projection map (or bundle projection). We shall assume in what follows that the base space B is connected.

We require that for every $x \in B$, there is an open neighborhood $U \subseteq B$ of x (which will be called a trivializing neighborhood) such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ (where $\pi^{-1}(U)$ is given the subspace topology, and $U \times F$ is the product space) in such a way that π agrees with the projection onto the first factor. That is, the following diagram should commute:

ADD THIS!

Denote the *tangent bundle*

$$TM = \cup_{x \in M} T_x M \qquad T_x M = \{m(t) \mid m(0) = x\}/\sim.$$

3.3 Lie Groups

Let TG be the tangent bundle to a Lie group G. We define

$$d(L_g)_h : T_h G \to T_{gh} G \quad \text{where} \quad h'(t) \mapsto (gh)'(t).$$

Notice that then

$$d(L_g)_1 : T_1 G \simeq T_g G.$$

Moreover,

$$G \times T_1 G \simeq TG \quad \text{where} \quad (g, v) \mapsto (g, d(L_g)_1 v).$$

Thus, TG is trivial as a vector bundle for a Lie group G. i.e. G is *parallelizable*.

3.4 Lie Algebra

Proposition 3.4.1.

$$[\phi * X, \phi * Y] = \phi * ([X, Y]).$$

Definition 3.4.2. Let G be a Lie group. A vector field X on G is said to be

- *right invariant* if, for every $g \in G, (R_g)_* X = X$. In detail,

  $$d(R_g)_h (X(h)) = X(hg)$$

  for every $g, h \in G$;
- *left invariant* if, for every $g \in G, (L_g)_* X = X$, that is,

  $$d(L_g)_h (X(h)) = X(gh).$$

Definition 3.4.3. We define *Maurer-Cartan forms*, which are differential 1forms on G with values in T_1G. They are defined by right or left translations by

$$\omega_g^r(v) = d\left(R_{g^{-1}}\right)_g (v) \quad \text{and} \quad \omega_g^l(v) = d\left(L_{g^{-1}}\right)_g (v)$$

for $g \in G$ and $v \in T_gG$.

Proposition 3.4.4. *If $X \in Vect(G)$ is right-invariant, then $\omega^r(X) = X(1)$, the constant T_1G-valued function. Similarly, if X is left-invariant, then $\omega^l(X) = X(1)$.*

Definition 3.4.5. We define the set of right invariant fields as

$$Inv_r = \cap_{g \in G} ker\left((R_g)_* - Id_{vect(G)}\right) \subseteq Vect(G).$$

Theorem 3.4.6. *Let $Inv_r \cong T_1G \cong Inv_e$*

Definition 3.4.7. $\mathfrak{g} = (Inv_r, [,])$ is the *Lie algebra* of a Lie group G.

Proposition 3.4.8. *This bracket gives the following bracket on T_1G:*

$$A \in T_1G \to A^r(g) = d(R_g)_1 A.$$

Moreover

$$[A, B] := [A, B]_r = [A^r, B^r](1).$$

Proposition 3.4.9. *Let $A, B \in T_1G$. Then, $[A, B]_r = -[A, B]_l$.*

$$[A, B] = -[A, B]_e = BA - AB.$$

3.5 Exponential Map

Remarks on flows on manifolds.

Let X be a vector field on manifold M, $X \in C^\infty(M, TM)$. A *flow* ϕ_t^x defined by $\phi_t^x(x) = x(t)$, $t \in (-\epsilon, \epsilon)$, and $\frac{dx}{dt} = X(x), x(0) = x$.

Another notation is $X_t = \phi_t^x$.

WTS

$$X_{s+t} = X_s \circ X_t = X_t \circ X_s.$$

Take $X \in \mathfrak{g} = Inv^r$ right invariant vector field

Then $X_t(g)$ the flow equals to $g(t)$ and is given by

$$\frac{dg}{dt} = X(g), \quad g(0) = g.$$

For $g \in G, g(t) : (-\epsilon, \epsilon) \to G$.

Lemma 3.5.1. *For $X \in Inv^r$, we have*

$$X_t(gh) = X_t(g)h \quad \forall g, h \in G.$$

Theorem 3.5.2. *A right-invariant vector field X is complete, i.e., defined for all $t \in \mathbb{R}$.*

G a lie group, $\mathfrak{g} = T_1G$ its lie algebra.

Definition 3.5.3. The *exponential map*

$$\exp : \mathfrak{g} \to G$$

is defined by $X \in \mathfrak{g}$ generates the right invariant vector field $X^r(g) = d(R_g)_1 X, g \in G$.

Then we create a flow, denoted by $X_t^r = g(t)$, for $\frac{dg(t)}{gt} = X^r(g(t)), g(0) = g$, which gives that $X_t^r(1) \mid_{t=1} = \exp(X)$.

Proposition 3.5.4. *By doing the same procedure using left-invariant vector field X^l gives the same result:*

$$X_t^l(1) \mid_{t=1} = X_t^r(1) \mid_{t=1} = \exp(X).$$

Moreover,

$$X_t^l(1) = X_t^r(1) \quad \forall t \in \mathbb{R}.$$

Proof. Denote $g(t_0) = X_t^r(1), g(0) = 1$. It's sufficient to show that $\frac{dg}{dt} = X^l(g)$.

We know that

$$
\begin{aligned}
\frac{dg}{dt} &= \frac{d}{dt}\left(X^r_t(1)\right) = \frac{d}{ds}\left(X^r_{s+t}(1)\right)|_{s=0} \\
&= \frac{d}{ds}\left(X^r_t\left(X^r_s(1)\right)\right)|_{s=0} \\
&= \frac{d}{ds}\left(X^r_t(1)X^r_s(1)\right)|_{s=0} \\
&= \frac{d}{ds}\left(L_{X^r_t(1)}X^r_s(1)\right)|_{s=0} \\
&= d(L_{X^r_t(1)})_1\frac{d}{ds}\left(X^r_s(1)\right)|_{s=0} && \text{chain rule} \\
&= d(L_{X^r_t(1)})_1 X^r(1) \\
&= d(L_{X^r_t(1)})_1 X \\
&= X^l(X^r_t(1)) \\
&= X^l(g(t))
\end{aligned}
$$

□

We have

$$X_t(1) : (\mathbb{R}, t) \to G.$$

a homomorphism, sometimes we call it a *one-parametric* subgroup of G generated by a right invariant vector field X^r.

Q: What is $X^r_t(1)$ and $X^l_t(1)$ via exp?

A: Suppose Y a vector field on M. Suppose we run a corresponding flow Y_t on M. Let $a \in \mathbb{R}$, then $(aY)_t = Y_{at}$ whenever flow Y_{at} and Y_t are defined.

$$(tY)_s|_{s=1} = Y_t.$$

APplying this to $M = G, Y = X^r$ at $g = 1, tX^r = (tX)^r$, we have

$$\exp(tX) = (tX)^r_s(1)|_{s=1} = (tX^r)_s(1)|_{s=1} = X^r_t(1).$$

Then

$$X^r_t(1) = \exp(tX) \quad X^l_t(1) = \exp(tX).$$

From office hour: $(\phi_* X)(y) = (d\phi)_{\phi^{-1}(y)}X(\phi^{-1}(y))$ pushforward

3.6 Exponential Map Formulas

One formula is that

$$\exp((s+t)X) = \exp(sX)\exp(tX) = \exp(tX)\exp(sX), \quad \forall x, t \in \mathbb{R}, x \in \mathfrak{g}.$$

This implies that for all X,

$$\{\exp(tX) \mid t \in \mathbb{R}\}$$

is an abelian subgroup of G.

Take $X \in \mathfrak{g}$, $X^r \in Inv^r$, $g \in \mathfrak{g}$, we have

$$X_t^r(g) = X_t^r(1)g \text{ because } X_t^r(gh) = X_t^r(g) = h.$$

This implies that

$$X_t^r(g) = X_t^r(1)g = \exp(tX)g, \text{ similarly } X_t^l(g) = g\exp(tX).$$

We also have

1. $\exp(0) = 1$;
2. $\exp(nX) = \exp(X)^n$ for all $n \in \mathbb{Z}$;
3. $\exp(X)^{-1} = \exp(-X)$.

Note that $\mathfrak{g} \cong \mathbb{R}^N$, so $T_y\mathfrak{g} = \mathfrak{g}$ for all $y \in \mathfrak{g}$.

Proposition 3.6.1. *$\exp : \mathfrak{g} \to G$ is smooth, and*

$$d(\exp)_0 : T_0\mathfrak{g} \to T_1 G \quad \textit{where} \quad X \mapsto X.$$

In other words, $d(\exp)_0 = \mathrm{id}_{\mathfrak{g}}$.

Proof. $\exp(X)$ is smooth because $X^r(g) = d(R_g)_1 X$ depends smoothly on X. Then flow $X_t^r(g)$ depends smoothly on X^r. Thus specialization of $X_t^r(g)$ at $g = 1, t = 1$ is also smooth as a function of X. Thus

$$\exp(X) = X^r(1)|_{t=1}$$

is smooth.

Now let's compute the differential.

$$\begin{aligned} d(\exp)_0(X) &= \frac{d}{dt}\left(\exp(0 + tX)\right)|_{t=0} \\ &= \frac{d}{dt}\left(X_t^r(1)\right)_{t=0} \\ &= X^r(1) \\ &= X \end{aligned}$$

□

By inverse function theorem, $\exp : \mathfrak{g} \to G$ is a diffeomorphism locally near $0 \in \mathfrak{g}$, i.e. there is an open neighborhood $U \subseteq \mathfrak{g}$ of 0 and an open neighborhood $V \subseteq G$ of 1 such that

$$\exp|_U : U \to V$$

is a diffeo-morphism.

Theorem 3.6.2. *If G is connected, then for all $g \in G$, there exists $x_1, \ldots, x_n \in \mathfrak{g}$ such that $g = \exp(x_1)\cdots\exp(x_n)$.*

Proof. Let G be a connected topological group, V any open neighborhood of 1. Then $G = \cup_{n\geq 1} V^n$. For all $g \in G$, there exists n such that $g \in V^n$. In other words, $g = v_1 \cdots v_n$ where $v_i \in V$.

Take V from the previous remark about exp a locally diffeomorphism locally near 0, we have $v_i = \exp(x_i)$ for some $x_i \in U$. □

3.7 Lie Algebras and Lie Group Homomorphisms

Let G, H be Lie groups. A *Lie group homomorphism* $\phi : G \to H$ is a smooth map which is a group homomorphism.

We claim that for a group homomorphism $\phi : G \to H$. For ϕ to be a Lie group homomorphism, it's enough to check the differentiability just at $g = 1$.

Notice that

$$\phi = R_{\phi(g)} \circ \phi \circ R_{g^{-1}}.$$

For h close to g in G, we have

$$\phi(h) = (R_{\phi(g)} \circ \phi)(hg^{-1}).$$

Therefore, $(d\phi)_1$ exists implies $d(R_{\phi(g)} \circ \phi)_1$ exists, and then $(d\phi)_g$ exists.

Proposition 3.7.1 (Lemma 5.14)**.** *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let $\phi : G \to H$ be a differentiable homomorphism and take $X \in \mathfrak{g}$. Then, for every $g \in G$, it holds*

$$d\phi_g\left(X^r(g)\right) = Y^r(\phi(g)) \quad d\phi_g\left(X^l(g)\right) = Y^l(\phi(g)),$$

where $Y = d\phi_1(X)$.

This proposition shows that X^r and Y^r (same with X^l and Y^l) are *ϕ-related*, i.e. $d\phi_x(X(x)) = Y(\phi(x))$.

Proposition 3.7.2. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let $\phi : G \to H$ be a differentiable homomorphism and take $X \in \mathfrak{g}$. Then,*

$$\phi(\exp(X)) = \exp\left(d\phi_1(X)\right).$$

Proposition 3.7.3 (Proposition A.2)**.** *Let $\phi : M \to N$ be a differentiable map and X_1, X_2 vector fields on M. Suppose that Y_1 and Y_2 are vector fields on N that are ϕ-related to X_1 and X_2, respectively. Then $[X_1, X_2]$ and $[Y_1, Y_2]$ are ϕ-related.*

Proposition 3.7.4 (Proposition 5.16)**.** *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let $\phi : G \to H$ be a differentiable homomorphism. Then, $d\phi_1 : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism, that is,*

$$d\phi_1[X, Y] = [d\phi_1 X, d\phi_1 Y]$$

with left or right invariant brackets.

Example 3.7.5. *Consider*

$$\det : GL(n, \mathbb{R}) \to \mathbb{R}^\times = GL(1, \mathbb{R}).$$

Then we know

$$d(\det)_1 : \mathfrak{gl}(n, \mathbb{R}) \to \mathbb{R}.$$

Proposition 3.7.6. *From the above example, we have*

$$d(\det)_1 A = \operatorname{tr} A.$$

Proof. We have $G = GL(n,\mathbb{R}), A \in T_1 G$. Consider

$$\alpha(t) : (-\epsilon, \epsilon) \to G \quad \text{where} \quad \alpha(0) = 1, \alpha'(0) = A.$$

Then

$$\begin{aligned}
d(\det)_1 A &= \frac{d}{dt}(I_n + tA)|_{t=0} \\
&= \frac{d}{dt}\left(t^n \chi_{-A}\left(\frac{1}{t}\right)\right)|_{t=0} \\
\chi_A(\lambda) &= \det(\lambda I_n - A) \\
&= (\lambda - \lambda_1)\cdots(\lambda - \lambda_n) \\
&= \lambda^n - (\operatorname{tr} A)\lambda^{n-1} + \cdots + (-1)^n \det A \\
&= \frac{d}{dt}\left(1 + t(\operatorname{tr} A) + \cdots + t^n \det A\right)|_{t=0} \\
&= \operatorname{tr} A
\end{aligned}$$

□

Remark that

$$\ker \det = \{g \in GL(n,\mathbb{R}) \mid \det g = 1\} = SL(n,\mathbb{R}).$$

3.8 The Adjoint Representation

Definition 3.8.1. A *representation of a Lie group* G on a finite vector space V is a Lie group homomorphism

$$\rho : G \to GL(V) \cong GL(n,\mathbb{R}).$$

Example 3.8.2 (Martin Page 105)**.** *Let $G = \mathrm{Gl}(n,\mathbb{R})$. Its canonical representation on $\mathbb{R}^n$ is the identity map. The corresponding infinitesimal representation is also the identity, that is, it associates with an element of $\mathfrak{gl}(n,\mathbb{R})$ the corresponding linear map of $\mathbb{R}^n$. This statement follows from*

$$\frac{d}{dt}\left(e^{tA}\right)_{|t=0} = A$$

Example 3.8.3 (Martin Page 106)**.** *Again, let $G = \mathrm{Gl}(n,\mathbb{R})$ and consider the tensor product*

$$T_k = \bigotimes^k \mathbb{R}^n = \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n.$$

For $g \in G$, define the linear map $\rho_k(g) : T_k \to T_k$ in such a way that, for the tensor products $v_1 \otimes \cdots \otimes v_k, v_1, \ldots, v_k \in \mathbb{R}^n$, it holds

$$\rho_k(g)\left(v_1 \otimes \cdots \otimes v_k\right) = gv_1 \otimes \cdots \otimes gv_k.$$

Map ρ_k is a representation of $\mathrm{Gl}(n,\mathbb{R})$. Its infinitesimal representation is computed with the derivative

$$\frac{d}{dt}\left(e^{tA}v_1 \otimes \cdots \otimes e^{tA}v_k\right)_{|t=0} = \sum_{i=1}^k v_1 \otimes \cdots \otimes Av_i \otimes \cdots \otimes v_k$$

The right hand side in this equality defines the linear map $(d\rho_k)_1(A)$. The tensor representation can be restricted to any linear group $G \subset \mathrm{Gl}(n, \mathbb{R})$.

Analogous representations are obtained for the k-th exterior product $\wedge^k \mathbb{R}^n$. The expressions for $\rho_k(g)$ and $(d\rho_k)_1$ are the same, replacing the tensor product $\otimes$ by the exterior product $\wedge$.

Definition 3.8.4. The *adjoint representation* $\mathrm{Ad} : G \to \mathrm{Gl}(\mathfrak{g})$, of G on its Lie algebra $\mathfrak{g}$ is defined by

$$\begin{aligned}\mathrm{Ad}(g) &= d\left(C_g\right)_1 = d\left(L_g \circ R_{g^{-1}}\right)_1 = d\left(R_{g^{-1}} \circ L_g\right)_1 \\ &= \left(dL_g\right)_{g^{-1}} \circ \left(dR_{g^{-1}}\right)_1 = \left(dR_{g^{-1}}\right)_g \circ \left(dL_g\right)_1.\end{aligned}$$

The representation Ad is differentiable.

Recall

$$d(Ad)_1 = \mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \quad \text{where} \quad X \mapsto [X, -].$$

Corollary 3.8.5 (Proposition 5.19)**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$, with bracket given by left invariant vector fields. Then, $d(\mathrm{Ad})_1(X) = \mathrm{ad}_l(X)$ for every $X \in \mathfrak{g}$ and*

$$\mathrm{Ad}(\exp X) = \exp\left(\mathrm{ad}_l(X)\right)$$

Proposition 3.8.6. *If G is abelian, then $\mathfrak{g}$ is abelian. If $G = G_0$, then $\mathfrak{g}$ abelian which implies G is abelian.*

Proposition 3.8.7. *We have* $\ker \mathrm{Ad} = \mathrm{Ad}^{-1} \subseteq G$ *(closed subgroup). And* $\ker \mathrm{Ad} = Z(G_0)$ *(centralizer of G_0).*

3.9 Haar Measure on Lie Group

Definition 3.9.1. A left(right) *Haar measure* is a measure invariant under left (right) translations.

$$\omega \in \Omega^n(G)$$

invariant under left translation gives a Haar measure. It means that

$$L_g^*(\omega) = \omega \quad \forall g \in G.$$

Example 3.9.2. *Let $G = GL(n, \mathbb{R})$. The Haar measure would be*

$$\frac{1}{(\det g)^n} \wedge dg_{ij} \quad g \in GL(n, \mathbb{R}).$$

We have ω is left-invariant iff for all g, h,

$$((L_g)^*\omega)(h) = \omega(h) \quad \text{i.e.} \quad (L_{g^{-1}})_g^*\omega(1) = \omega(g).$$

4 Lie's Fundamental Theorem

4.1

Theorem 4.1.1 (Lie's Third and Second Theorem)**.** *The functor from simply connected Lie group to Lie algebra establishes an equivalence of categories. (it's surjective)*

Proposition 4.1.2. *For every finite dimensional Lie algebra $\mathfrak{g}$, there exists a Lie group with $\mathfrak{g}$ as its Lie algebra.*

In mathematics, the *Baker-Campbell-Hausdorff formula* is the solution for Z to the equation

$$e^X e^Y = e^Z$$

for possibly noncommutative X and Y in the Lie algebra of a Lie group. There are various ways of writing the formula, but all ultimately yield an expressio for Z in Lie algebraic terms, that is, as a formal series (not necessarily convergent) in X and Y and iterated commutators thereof. The first few terms of this series are:

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \cdots$$

4.2 Baker Campbell Hausdorff Theorem

Let $\mathbb{R}\langle x, y\rangle$ be the free associated algebra on x, y, also could be written as the tensor algebra of $V = \mathbb{R}x \oplus \mathbb{R}y$, written as $T(V) = \oplus_{n\geq 0} V^{\oplus n}$.

Given A associated $\mathbb{R}$-algebra. Denote $A[[s, t]]$ the algebra of formal power series in $s, t (st = ts)$, could be written as

$$A[[s, t]] = \{a_{00} + a_{10}s + a_{01}t + a_{11}st + a_{12}st^2 + a_{21}s^2t + \cdots \mid a_{ij} \in A\}$$

$$A = \lim_{\leftarrow} A[s, t]/(s, t)^n.$$

Define

$$\ell(xs, yt) = \log(\exp(xs)\exp(yt)) = \log(e^{xs}e^{yt}),$$

where

$$e^{xs} = 1 + \frac{xs}{1!} + \frac{x^2s^2}{2!} + \frac{x^3s^3}{3!} + \cdots$$

and

$$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots .$$

$z \in (s, t) \subseteq \mathbb{R} < x, y > [[s, t]]$. And $\log(\alpha) = \log(a + (\alpha - 1))$, provided $\alpha - 1 \in (s, t)$.

Theorem 4.2.1 (BCH)**.** 1. *For the formal power series $\ell(xs, yt)$, we have*

$$\ell(xs, yt) = xs + yt + \frac{1}{2}[x, y]st + \frac{1}{12}[x, [x, y]]s^2t + \frac{1}{12}[y, [y, x]]st^2 + \cdots$$

with all the coefficients in power series $\ell(xs, yt)$ given by Lie-bracket polynomials, where $[x, y] := xy - yx \in \mathbb{R}\langle x, y\rangle$. The coefficients may be obtained by a recursive formula.

2. *Given a Lie group G, there exists $u' \subseteq u \subseteq \mathfrak{g}$ and $V \subseteq G$ (open neighborhoods of 0 and 1, resp.) such that $\exp(\mathfrak{g}) = G$ and $\log(V) = u$. And u' is such that for all $X, Y \in u'$, we have $\exp(X)\exp(Y) \in V$, which allows us to apply $\log$:*

$$C(X,Y) := \log(\exp(X)\exp(Y)).$$

Then the series $\ell(X,Y)$ as a series in $\mathfrak{g}$, converges to $C(X,Y)$.

Corollary 4.2.2. *A smooth Lie group G is real analytic.*

4.3 Universal Enveloping Algebra

Let V be a vector space with $k = \mathbb{R}, \mathbb{C}, \mathbb{Q}_p$ or any field of

Let $T(V)$ be the free algebra generated by V. Universal property iff Functor $T : Vect \to Alg$ from vector space to associated algebras is a left forgetful functor:

$$F : Alg \to Vect \quad A \mapsto F(A) = A.$$

That is a natural bijection

$$\mathrm{Hom}_{Alg}(T(V), A) \cong \mathrm{Hom}_{Vect}(V, A).$$

There exsits a left adjoint of $Alg \to LieAlg$ which takes A to $\mathfrak{g}(A) = A$, denoted by $U : LieAlg \to Alg$ where $\mathfrak{g} \mapsto U\mathfrak{g}$, which is called the *universal enveloping algebra*.

Definition 4.3.1. $U\mathfrak{g} = T(\mathfrak{g})/(xy - yx - [x,y])$ with $x, y \in \mathfrak{g}$.

Proposition 4.3.2. *This is a left-adjoint, indeed.*

4.4 Completing the Proof of PBW Theorem

Given a filtered vector space V, we define $\mathrm{gr}\, V \overset{\text{def}}{=} \bigoplus_{n \geq 0} \mathrm{gr}_n V$, where $\mathrm{gr}_n V \overset{\text{def}}{=} V_{\leq n}/V_{\leq (n-1)}$.

A Lie algebra $\mathfrak{g}$ is abelian if the bracket is identically 0 . If $\mathfrak{g}$ is abelian, then $\mathcal{U}\mathfrak{g} = \mathcal{S}\mathfrak{g}$, where $\mathcal{S}V$ is the symmetric algebra generated by the vector space V (so that $\mathcal{S}$ is left-adjoint to Forget: CoMALG $\to$ VECT).

Theorem 4.4.1 (Poincaré-Birkhoff-Witt)**.** *The map $\mathcal{S}\mathfrak{g} \to \mathrm{gr}\,\mathcal{U}\mathfrak{g}$ is an isomorphism of algebras.*

4.5 Bialgebra

Definition 4.5.1. An element x of a coalgebra B is called *primitive* if $\delta(x) = x \otimes 1 + 1 \otimes x$. Denote $prim(B)$ as the set of all primitive $x \in B$.

Proposition 4.5.2. *There is a bialgebra structure on $\mathcal{U}\mathfrak{g}$ for a Lie algebra $\mathfrak{g}$ such that $prim(\mathcal{U}\mathfrak{g}) = \mathfrak{g}$.*

4.6 Universal algebra as Differential Operators

Definition 4.6.1 (Grothendieck)**.** Let X be a space and $\mathscr{S}$ a sheaf of functions on X. We define the *sheaf $\mathscr{D}$ of Grothendieck differential operators* inductively. Given open $U \subseteq X$, we define $\mathscr{D} \leq 0(U) = \mathscr{S}(U)$, and

$$\mathscr{D}_{\leq n}(U) = \left\{x : \mathscr{S}(U) \to \mathscr{S}(U) \text{ s.t. } [x, f] \in \mathscr{D}_{\leq (n-1)}(U) \forall f \in \mathscr{S}(U)\right\},$$

where $\mathscr{S}(U) \curvearrowright \mathscr{S}(U)$ by left-multiplication. Then $\mathscr{D}(U) = \bigcup_{n \geq 0} \mathscr{D}_{\leq n}(U)$ is a filtered sheaf; we say that $x \in \mathscr{D}_{\leq n}(U)$ is an *nth-order differential operator on U*.

$\mathscr{D}_{\leq 1}$ is a subsheaf, a sheaf of Lie algebras.

By Jacobi identity we have

$$[\mathscr{D}_{\leq m}, \mathscr{D}_{\leq n}] \subseteq \mathscr{D}_{m+n-1}.$$

We have

$$\mathscr{D}_{\leq 1}(U) = C^\infty_M(U) \oplus Vect(U) = \{D \in \mathscr{D}_{\leq 1}(U) \mid D(1) = 0\}.$$

$\mathscr{D}$ is generated as a sheaf of associated algebras by C^∞_M and TM.

Proposition 4.6.2. *Let G be a Lie group, and $\mathscr{D}(G)^G$ the subalgebra of left-invariant differential operators on G. The natural map $\mathcal{U}\mathfrak{g} \to \mathscr{D}(G)^G$ generated by the identification of $\mathfrak{g}$ with leftinvariant vector fields is an isomorphism of algebras.*

Lemma 4.6.3. *Suppose for some $u \in \mathcal{U}\mathfrak{g} = \mathscr{D}(G)^G$, we have $(uf)(1) = 0$ for each $f \in C^\infty_{G,1}$ stalk of germs of functions C^∞_G at 1. Then $u = 0$.*

Proof. Let's show $u = 0$ as a differential operator. For all $g \in G$,

$$(uf)(g) = (L_g^*(uf))(1) = (u(L_g^* f))(1) = 0.$$

Thus $uf = 0$ and $u = 0$. □

Proposition 4.6.4. *Extend Δ to*

$$\Delta : \mathcal{U}\mathfrak{g}[[s]] \to (\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g})[[s]].$$

then Δ is an s-adic-continuous algebra homomorphism. Then $u(s) \in \mathcal{U}\mathfrak{g}[[s]]$ is primitive iff $e^{u(s)}$ is group-like, that is

$$\Delta(e^{u(s)}) = e^{u(s)} \otimes e^{u(s)},$$

.

4.7 Proof of BCH

Theorem 4.7.1 (BCH Formula). *1. Recall $\mathbb{R}\langle x, y\rangle = T(\mathbb{R}x \oplus \mathbb{R}_y) = \mathcal{U}Lie(x, y)$, the free Lie algebra on x, y. First define a series $b(sx, ty) \in \mathbb{R}\langle x, y\rangle[[s, t]]$ by $e^{b(sx,ty)} = e^{sx}e^{ty}$. Then $b(sx, ty) \in Lie(x, y)[[s, t]]$, i.e., $b(sx, ty) = \sum_{p,q\geq 0} b_{pq}(x, y)s^p t^q$ with $b_{pq}(x, y) \in Lie(x, y)$ and*

$$b(sx, ty) = xs + ty + \frac{1}{2}[x, y]st + \frac{1}{12}[x, [x, y]]s^2t + \frac{1}{12}[y, [y, x]]st^2 + \cdots$$

2. *Suppose G is a real (or complex) analytic Lie group (will be true for C^∞-Lie groups, see San Martin), then there exists open neighborhoods $0 \in U \subseteq \mathfrak{g}$ and $1 \in V \subseteq G$ such that*

$$\exp : U \to V \quad \textit{and} \quad \log : V \to U$$

establish a diffeomorphism and $b(X, Y)$, as a series $\sum b_{p,q}(X, Y)$ in $\mathfrak{g}$ converges on U to $\log(\exp(X)\exp(Y))$.

Lemma 4.7.2. *For all $X \in \mathfrak{g}, g \in G, s \in \mathbb{R}$, and f a real analytic function f on G, we have*

$$\left(e^{sX^e} f\right)(g) = f(g \exp(sX)).$$

4.8 Universal Covering Group

Theorem 4.8.1. *Given G connected Lie group, $\widetilde{G}$ its universal cover as a topological space, $p : \widetilde{G} \to G$.*

1. *Pick $\widetilde{1} \in \widetilde{G}$ such that $p(\widetilde{1}) = 1$. Then $\widetilde{G}$ has a unique structure of a Lie group such p is smooth covering and a Lie group homomorphism and $\widetilde{1} \in \widetilde{G}$ is the identity.*
2. *$\widetilde{G} \to G$ induces an isomorphism $\widetilde{\mathfrak{g}} \to \mathfrak{g}$.*
3. *Adjunction*

$$\mathrm{Hom}_{LieGrp}(\widetilde{G}, H) \cong \mathrm{Hom}_{LieAlg}(\mathfrak{g}, \mathfrak{h}),$$

where H is a Lie group with Lie algebra $\mathfrak{h}$.

Corollary 4.8.2. *For all finite dimensional $\mathfrak{g}$ Lie algebra, there exists a simply connected Lie group with Lie algebra $\mathfrak{g}$.*

5 Classification

5.1 Lie groups