Analysis 2 Functional Analysis

Yuxuan Sun

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1 Basics

Definition 1.1: characteristic function

Given $S \subset \mathbb{R}$, the corresponding characteristic function is

$$\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \in S^C \end{cases}$$

Definition 1.2: Lebesgue integral of step function

Given a step function

$$f(x) = c_1 \chi_{I_1}(x) + c_2 \chi_{I_2}(x) + \ldots + c_n \chi_{I_n}(x)$$

We define its Lebesgue integral to be

$$\int_{-m}^{\infty} f(x)dx = c_1 m\left(I_1\right) + c_2 m\left(I_2\right) + \cdots + c_n m\left(I_n\right)$$

2 L1

Definition 2.1: $L^1\mathbb{R}$

We say $f : \mathbb{R} \to \mathbb{R}$ is in $L^2(\mathbb{R})$ if there are functions $g, h \in L^0(\mathbb{R})$ s.t. $f = g - h$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} g(x)dx - \int_{-\infty}^{\infty} h(x)dx$$

Theorem 2.2: $L^1(\mathbb{R})$ is a vector space

If $f_1, f_2 \in L^1(\mathbb{R})$ and c_1, c_2 are real constancts, then $c_1 f_1 + c_2 f_2 \in L^1(\mathbb{R})$

Theorem 2.3: $L^1(\mathbb{R})$ order integral

If $f_1, f_2 \in L^1(\mathbb{R})$ and $f_1(x) \geq f_2(x)$ for all $x \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} f_1(x)dx \geq \int_{-\infty}^{\infty} f_2(x)dx.$$

Definition 2.4: $L^1_{nvs}(\mathbb{R})$

An element of $L^1_{nvs}(\mathbb{R})$ is a collection of functions in $L^1(\mathbb{R})$: specifically, two functions are in the same collection if they are equal except on a set of measure zero.
Given a collection S in $L^1_{nvs}(\mathbb{R})$, define $\|S\|_1$ by choosing any $f \in S$ and defining $\|S\|_1 = \|f\|_1$

Theorem 2.5: Monotone Convergence Theorem in L^1

Let $f_n \in L^1(\mathbb{R})$ which monotone increases for all $x \in \mathbb{R}$.
Suppose $\left\{\int_{-\infty}^{\infty} f_n(x)dx \; n \in \mathbb{N}\right\}$ is bounded.
Then there exists $f \in L^1(\mathbb{R})$ s.t. $f_n \to f$ pointwise except possibly on a set of measure zero, and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx$$

Theorem 2.6: $L^1([a, b])$ is complete

If $f_n \in L^1([a, b])$ is a Cauchy sequence (with resepct to $\|\cdot\|_1$). Then there exists $f \in L^1([a, b])$ s.t. $f_n \to f$ in L^1

Definition 2.7: the spaces $L^p(\mathbb{R})$

For $p > 1$, we say that $f \in L^p(\mathbb{R})$ if f is a measurable function and $\int_{-\infty}^{\infty} |f(x)|^p dx$ is a finite number.

Theorem 2.8: $L^p(\mathbb{R})$ is a vector space

$L^p(\mathbb{R})$ is a vector space.

3 $L^1([a, b])$ and Fundamental THeorems of Calculus

Definition 3.1

if $f : [a, b] \to \mathbb{R}$, we say that $f \in L^1([a, b])$ if the function

$$g(x) = \begin{cases} f(x) & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

is in $L^1(\mathbb{R})$. In that case, we write $\int_a^b f(x)dx = \int_{-\infty}^{\infty} g(x)dx$

4 L^2

Theorem 4.1: Inner Product on $L^2(\mathbb{R})$

if $f, g \in L^2$, then $fg \in L^1$ with:

$$\int_{-\infty}^{\infty} |f(x)g(x)| \leq \|f\|_2 \|g\|_2$$

Definition 4.2: Inner product on L^2

If $f, g \in L^2$, let:

$$\langle f, g \rangle - \int_{-\infty}^{\infty} f(x)g(x)dx$$

5 Geometry Recap

Theorem 5.1: Parallel-gram Law

d_1 and d_2 being the diagnoal an

$$s^2 + s^2 + t^2 + t^2 = d_1^2 + d_2^2$$

Corollary 5.2: inner product

$\|\cdot\|_2$ has an inner product that's like the dot product.
$\|\cdot\|_p$ and $\|\cdot\|_\infty$ don't have inner product.

6 Fourier

Example 6.1: orthonormal set

i.e. the inner product is 0

$$\frac{1}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{2\pi}}, \frac{\cos 2x}{\sqrt{\pi}}$$

restricted to domain $[-\pi, \pi]$

Definition 6.2: Fourier senes

Given $f \in L^2$, define its Fourier senes as:

$FS_f(x) = \langle f, f_0 \rangle f_0(x) + \langle f, f_1 \rangle f_1(x) + \langle f, f_2 \rangle f_2(x) + \langle f, f_3 \rangle f_3(x) + \dots$

Comments
values of f outside $[-\pi, \pi]$ have no impact on FS_f
we could assume $f = 0$ outside $[-\pi, \pi]$, i.e. $f \in L^2([-\pi, \pi))$

Definition 6.3: inner product in L^2

Given $f, g \in L^2(\mathbb{R})$, we define there inner product by:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx$$

Theorem 6.4

If $f \in L^2([-\pi, \pi])$, $FS_f \to f$ in L^2

Theorem 6.5

If $f \in C([-\pi, \pi])$ and $f(\pi) = f(-\pi)$, then $FS_f \to f$ uniformly on $[-\pi, \pi]$

7 Fourier transform

Definition 7.1: rapidly decresing

f is rapidly decreasing if, for any $n \in \mathbb{N}$, there exists M_n, C_n s.t. $|f(x)| \le C_n/x^n$ for all x with $|x| > M_n$. An alternative perspective: f is rapidly decreasing if and only if for any polynomial $p(x)$, we have $\lim_{x\to\infty} p(x)f(x) = \lim_{x\to-\infty} p(x)f(x) = 0$.

Definition 7.2: $C^\infty(\mathbb{R})$

$f \in C^\infty(\mathbb{R})$ is f has infinitely many derivatives at all $x \in \mathbb{R}$

Definition 7.3: $S(\mathbb{R})$

$f \in S(\mathbb{R})$ (Schwartz-class) if f is rapidly decreasing and in $C^\infty(\mathbb{R})$

$e^{-x^2} \in S(\mathbb{R})$

Definition 7.4: $\mathcal{D}([a,b])$

$f \in \mathcal{D}([a,b])$ if $C \in C^\infty(\mathbb{R})$ and $f = 0$ outside of $[a,b]$

Theorem 7.5: L^2 and fourier

if $f \in L^2([-T,T])$, then

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2T} C_n e^{-in\pi x/T}$$

for

$$C_n = \int_{-T}^{T} f(x) e^{in\pi x/T}$$

7.1 crash course in $\mathbb{C}$

Given a complex-valued function

$$f(x) = f_1(x) + i f_2(x)$$

we would say $f \in L^1(\mathbb{R}), f_1, f_2 \in L^1(\mathbb{R})$ and its integral to be

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f_1(x)dx + i \int_{-\infty}^{\infty} f_2(x)dx$$

Theorem 7.6: Dominated Convergence Theorem

Given $f_n \in L^1(\mathbb{R})$ s.t.

- $f_n \to f$ except possibly on a set of measure zero
- $|f_n(x)| \leq g(x)$ for a function $g \in L^1(\mathbb{R})$

Then $f \in L^1(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx$$

7.2 Fourier Transform Properties

Corollary 7.7: addition

For $\tau_y : S(\mathbb{R}) \to S(\mathbb{R})$ defined by $(\tau_y f)(x) = f(x+y)$ if $f \in S(\mathbb{R})$, then

$$\widehat{(\tau_y f)}(\omega) = e^{-i\omega y}\hat{f}(\omega)$$

Let

$$f_n(x) = \begin{cases} f(x+y)e^{i\omega x} & -n \le x \le n \\ 0 & \text{otherwise} \end{cases}$$

Then $f_n \to f(x+y)e^{i\omega x}$ pointwise on $\mathbb{R}$, and

$$|f_n(x)| \le \left|f(x+y)e^{i\omega x}\right| \le |f(x+y)| \cdot 1$$

and

$$|f(x+y)| \in L^1(\mathbb{R})$$

because $f \in S(\mathbb{R})$ its translation is rapidly decreasing as well, so

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x+y)e^{i\omega x}dx &= \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x)dx \\
&= \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx \\
&= \lim_{n\to\infty} \int_{-\infty}^{\infty} f(x+y)e^{i\omega x}dx \\
&= \lim_{n\to\infty} \int_{-n-y}^{n-y} f(u)e^{i\omega(u-y)}du \\
&= e^{i\omega y} \lim_{n\to\infty} \int_{-n-y}^{n-y} f(u)e^{i\omega u}du \\
&= e^{i\omega y} \int_{-\infty}^{\infty} f(u)e^{i\omega u}du
\end{aligned}$$

Corollary 7.8: derivative

If $f \in S(\mathbb{R})$, then

$$\widehat{\left(i\frac{df}{dx}\right)}(\omega) = \omega\hat{f}(\omega)$$

Corollary 7.9: derivative 2

if $f \in S(\mathbb{R})$

$$\widehat{(xf)}(\omega) = -i\frac{d\hat{f}}{d\omega}(\omega)$$

Definition 7.10: convolution

Given $f, g \in \mathbb{R} \to \mathbb{R}$, their convolution is the function:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

(for any x such taht the integral exists).

Corollary 7.11

Given $y \in \mathbb{R}$, if $f \in S(\mathbb{R})$, then

$$\left(\widehat{e^{ixy}f}\right)(\omega) = \widehat{(}f)\,(\omega + y)$$

Corollary 7.12: Property 2

if $f \in S(\mathbb{R})$, then

$$\left(\widehat{i\frac{df}{dx}}\right)(\omega) = \omega\hat{f}(\omega)$$

Corollary 7.13: Property 2a

if $f \in S(\mathbb{R})$, then

$$(\widehat{xf})(\omega) = -i\frac{d\hat{f}}{d\omega}(\omega)$$

Proof

$$\begin{aligned}
\text{RHS} &= -i\frac{d}{dw}\left(\int_{-\infty}^{\infty} f(x)e^{iwx}dx\right) \\
&= -i\int_{-\infty}^{\infty} \frac{\partial}{\partial e}\left(f(x)e^{iwx}\right)dx \\
&= -i\int_{-\infty}^{\infty} f(x)e^{iwx}ixdx \quad \text{chain rule} \\
&= (-i)i\int_{-\infty}^{\infty} (xf(x))\,e^{iwx}dx \\
&= \widehat{xf}(w)
\end{aligned}$$

Corollary 7.14: Property 3

if $f, g \in S(\mathbb{R})$, then

$$\left(\widehat{f * g}\right)(\omega) = \hat{(}f)(\omega)\hat{(}g)(\omega)$$

Corollary 7.15: Property 3a

if $f, g \in S(\mathbb{R})$, then

$$\widehat{(fg)}(\omega) = \frac{1}{2\pi}\left(\hat{f} * \hat{g}\right)(\omega)$$

Theorem 7.16: Lebiniz Rule

if $f, \frac{\partial f}{\partial y}$ are continuous on $[a, b] \times [c, d]$, the following holds for $y \in [c, d]$

$$\frac{d}{dy}\int_a^b f(x, y)dx = \int_a^b \frac{\partial f}{\partial y}(x, y)dx$$

There's a version of the Leibniz Rule that incorporates the Fundamental Theorem:

$$\frac{d}{dy}\int_{a(y)}^{b(y)} f(x, y)dx = \int_{a(y)}^{b(y)} \frac{\partial f}{\partial y}(x, y)dx + f(b(y), y)\frac{db}{dy} - f(a(y), y)\frac{da}{dy}$$

Definition 7.17: contour integral

Given a function $f : \mathbb{C} \to \mathbb{C}$ and a path γ in the plane parametrized as

$$\vec{r}(t) = (x(t), y(t))$$

reinterpret it as a path in $\mathbb{C}$:

$$z(t) = x(t) + iy(t)$$

Then the integral of f ove the path γ is defined to be

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r_1 **integral**
parametrize: $z(t) = t, z'(t) = 1$

$$\int_{r_1} f(z)dz = \int_{-R}^{R} e^{-Bt^2}(1)dt$$

r_3 **integral**
parametrize $z(t) = -t + \frac{w}{2B}i$ for $-R < t < R$

$$\int_{r_3} f(z)dz = \int_{-R}^{R} e^{-B(-t+\frac{w}{2B}i)^2}(-1)dt$$

r_2 **integral**
parametrize $z(t) = R + ti$ for $0 \le t \le \frac{w}{2B}$

$$\int_{r_2} f(z)dz = \int_0^{\frac{w}{2B}} e^{-B(R+ti)^2} idt$$

Theorem 7.18: contour-integration theorem

e $f(z)$ is differentiable at every point inside and on a closed path γ (that's a path that returns to where it started), then the integral of f ove that closed path is zero.

Here are our Fourier Transform properties so far:

Function	Fourier Transform
$f(x+y)$	$e^{-i\omega y}\widehat{f}(\omega)$
$e^{ixy}f(x)$	$\widehat{f}(\omega+y)$
$if'(x)$	$\omega\widehat{f}(\omega)$
$xf(x)$	$-i(\widehat{f})'(\omega)$
$(f*g)(x)$	$\widehat{f}(\omega)\widehat{g}(\omega)$
$f(x)g(x)$	$\frac{1}{2\pi}(\widehat{f}*\widehat{g})(\omega)$
Ae^{-Bx^2}	$A\sqrt{\frac{\pi}{B}}e^{-\omega^2/(4B)}$

Definition 7.19: δ-sequence

We call $f_n \in L^1(\mathbb{R})$ a δ-sequence if
$\int_{-\infty}^{\infty} f_n(x)dx = 1$ for all n
for all $r > 0$, we have

$$\lim_{n\to\infty}\left(\int_{-r}^{r} f_n(x)dx\right) = 1$$

In other words,

$$\lim_{n\to\infty}\left(\int_{-\infty}^{r} f_n(x)dx + \int_{r}^{\infty} f_n(x)dx\right) = 0$$

8 distribution

Definition 8.1: linear operator

If $T : V \to W$ for V, W function spaces, and $T(c_1f_1 + c_2f_2) = c_1T(f_1) + c_2T(f_2)$ for all $f_1, f_2 \in V$ and $c_1, c_2 \in \mathbb{R}$. Then T is a **linear operator**.

Definition 8.2: adjoint

Suppose that $T : D(\mathbb{R}) \to D(\mathbb{R})$ is a linear operator. We say a linear operator S on $D(\mathbb{R})$ is the adjoint of T if

$$\int_{-\infty}^{\infty}(T\psi)(x)\varphi(x)dx = \int_{-\infty}^{\infty}\psi(x)(S\varphi)(x)dx$$

Definition 8.3: tempered distribution

A **tempered distribution** is a continuous linear functional on $S(\mathbb{R})$