

1 The Real Numbers

1.2 Some Preliminaries

Triangle Inequality
The absolute value function is an important tool in the study of the real numbers. It is defined for every real number x as follows: $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. With respect to multiplication and division, the absolute value function satisfies $||x| + y| = |x + y|$ and $||x| - y| = ||x| - y|$.

Theorem 1.2.6
Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.

1.3 The Axiom of Completeness

Axiom of Completeness
Every nonempty set of real numbers that is bounded above has a least upper bound.

Lemma 1.3.8
Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $\sup A = s$ if and only if for every choice of $\epsilon > 0$, there exists an element $x \in A$ satisfying $x > s - \epsilon$.

Definition 1.3.1
A set $A \subseteq \mathbb{R}$ is **bounded above** if there exists a number $b \in \mathbb{R}$ such that $x \leq b$ for all $x \in A$. The number b is called an **upper bound** for A . Similarly, the set A is **bounded below** if there exists a **lower bound** $c \in \mathbb{R}$ satisfying $c \leq x$ for every $x \in A$.

Definition 1.3.2
Definition 1.3.2. A real number s is the **least upper bound** for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:
1. s is an upper bound for A .
2. If t is any upper bound for A , then $s \leq t$.

1.4 Consequences of Completeness

Theorem 1.4.1 (Nested Interval Property)
For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n]$ in \mathbb{R} , $n \in \mathbb{N}$, such that $I_{n+1} \subseteq I_n$. Assume also that each I_n has a length $|I_n| = b_n - a_n < 1/n$. Then, the resulting nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ has a nonempty intersection that is, $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$.

Theorem 1.4.2 (Archimedean Property)
Theorem 1.4.2 (Archimedean Property).
(i) Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.
(ii) Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$.

Theorem 1.4.3 (Density of \mathbb{Q} in \mathbb{R})
For every two real numbers $a < b$, there exists a rational number r satisfying $a < r < b$.

Corollary 1.4.4
Given any two real numbers $a < b$, there exists an irrational number x satisfying $a < x < b$.

Theorem 1.4.5
There exists a real number $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.

1.5 Cardinality

Definition 1.5.1
A function $f: A \rightarrow B$ is **one-to-one** if $f(a) = f(b) \implies a = b$. The function f is **onto** if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Theorem 1.5.7
If $A \subseteq B$ and B is countable, then A is either countable or finite.

Theorem 1.5.8
(i) If $A, B \subseteq \mathbb{R}$ are each countable sets, then the union $A \cup B$ is countable.
(ii) If $A, B \subseteq \mathbb{R}$ are each countable sets, then the intersection $A \cap B$ is countable.

Definition 1.5.2
The set \mathbb{N} has the **same cardinality** as \mathbb{Z} if there exists $f: \mathbb{N} \rightarrow \mathbb{Z}$ and $g: \mathbb{Z} \rightarrow \mathbb{N}$ such that $f \circ g = \text{id}_{\mathbb{Z}}$ and $g \circ f = \text{id}_{\mathbb{N}}$. In this case, we write $\mathbb{N} \sim \mathbb{Z}$.

Definition 1.5.5
A set A is **countable** if $\mathbb{N} \sim A$. An infinite set that is not countable is called an **uncountable set**.

Theorem 1.6.1
The open interval $(0, 1) \subseteq \mathbb{R}$ is uncountable.

Theorem 1.6.2 Cantor's Theorem
Given any set A , there does not exist a function $f: A \rightarrow \mathcal{P}(A)$ that is onto.

Exercise 1.6.1
 $(0, 1)$ is uncountable if and only if \mathbb{R} is uncountable.

3 Basic Topology of \mathbb{R}

3.2 Open and Closed Sets

open
A set $O \subseteq \mathbb{R}$ is open for all points $x \in O$ there exists an open neighborhood $N_\epsilon(x) \subseteq O$.

limit point
A point x is a limit point of a set A if every open neighborhood $N_\epsilon(x)$ of x contains at least one point other than x .

isolated point
A point $x \in A$ is an isolated point of A if x is not a limit point of A .

closed
A set $C \subseteq \mathbb{R}$ is closed if it contains its limit points.

closure
Given a set $A \subseteq \mathbb{R}$, let \bar{A} be the set of all limit points of A . The **closure** of A is defined to be $\bar{A} \cup A$.

Theorem 3.2.14
The union of a finite collection of closed sets is closed.
The intersection of an arbitrary collection of closed sets is closed.

Theorem 3.2.3
The union of an arbitrary collection of open sets is open.
The intersection of a finite collection of open sets is open.

Theorem 3.2.13
A set O is open if and only if O is the union of open intervals.

3.3 Compact Sets

compact
A set $K \subseteq \mathbb{R}$ is compact if every sequence in K has a subsequence that converges to a limit that is also in K .

Heine-Borel Theorem
Let K be a subset of \mathbb{R} . All of the following statements are equivalent in the sense that any one of them implies the two others:
1. K is compact.
2. K is closed and bounded.
3. Every open cover for K has a finite subcover.

Characterization of Compactness in \mathbb{R}
Let $K \subseteq \mathbb{R}$. Then K is compact if and only if it is closed and bounded.

3.4 Perfect Sets and Connected Sets

perfect
A set $P \subseteq \mathbb{R}$ is perfect if it is closed and contains no isolated points.

separated
Two nonempty sets $A, B \subseteq \mathbb{R}$ are separated if $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$.

disconnected
A set $E \subseteq \mathbb{R}$ is disconnected if it can be written as $E = A \cup B$, where A and B are nonempty separated sets.

Theorem 3.4.6
A set $E \subseteq \mathbb{R}$ is connected if and only if for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n)_{n \in \mathbb{N}}$ contained in one of A or B , and x is an element of the other.

Theorem 3.4.7
A set $E \subseteq \mathbb{R}$ is connected if and only if whenever $a < x < b$ with $a, b \in E$, it follows that $x \in E$ as well.

Theorem 3.4.3
Theorem 3.4.3. A nonempty perfect set is uncountable.

7 The Riemann Integral

7.6 Lebesgue's Criterion for Riemann Integrability

measure zero
A set $A \subseteq \mathbb{R}$ has measure zero if, for all $\epsilon > 0$, there exists a countable collection of open intervals O_n such that $A \subseteq \bigcup_{n \in \mathbb{N}} O_n$ and the sum of the lengths of all the intervals is less than or equal to ϵ . More precisely, if O_n refers to the length of the interval O_n , then we have

Definition 8.6.4 (algebraic approach)
A set S is a field if there exist two operations—addition $(+)$ and multiplication (\cdot) that satisfy the following list of conditions:

Definition 8.6.5
Definition 8.6.5. An **ordering** on a set S is a relation $<$ represented by $<$, with the following three properties:

2 Sequences and Series

2.2 The Limit of a Sequence

Definition 2.2.1
A sequence is a function whose domain is \mathbb{N} .

Convergence of a Sequence
A sequence (a_n) converges to a real number L if for every positive number ϵ , there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$, it follows that $|a_n - L| < \epsilon$.

Theorem 2.2.4
Given a real number $a \in \mathbb{R}$ and a positive number $\epsilon > 0$, the set $N_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$ is called the **neighborhood** of a .

Theorem 2.2.7 Uniqueness of Limits
The limit of a sequence, when it exists, must be unique.

2.3 The Algebraic and Order Limit Theorems

Definition 2.3.1
A sequence (a_n) is bounded if there exists a number $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem 2.3.2
Every convergent sequence is bounded.

Theorem 2.3.3
(Algebraic Limit Theorem).
 $\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$

Squeeze Theorem
Show that if $a_n \leq p_n \leq b_n$ for all $n \in \mathbb{N}$, and if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = L$, then $\lim_{n \rightarrow \infty} p_n = L$ as well.

Caesar Means
Show that if (a_n) is a convergent sequence, then the sequence given by the averages

2.5 Subsequences and the Bolzano-Weierstrass Theorem

subsequence
Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < \dots$ be an increasing sequence of natural numbers. Then the sequence $(a_{n_k})_{k \in \mathbb{N}}$ is called a **subsequence** of (a_n) and is denoted by (a_{n_k}) , where $k \in \mathbb{N}$ indexes the subsequence.

Bolzano-Weierstrass Theorem
Every bounded sequence contains a convergent subsequence.

Theorem 2.5.2
Subsequences of a convergent sequence converge to the same limit as the original sequence.

2.6 The Cauchy Criterion

Cauchy sequence
A sequence (a_n) is called a **Cauchy sequence** if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that $|a_m - a_n| < \epsilon$.

Theorem 2.6.4 (Cauchy Criterion)
A sequence converges if and only if it is a Cauchy sequence.

Theorem 2.6.2
Every convergent sequence is a Cauchy sequence.

Lemma 2.6.3
Cauchy sequences are bounded.

4 Functional Limits and Continuity

4.2 Functional Limits

Definition 4.2.1 (Functional Limit)
Let $f: A \rightarrow \mathbb{R}$ and let c be a limit point of the domain A . We say that $\lim_{x \rightarrow c} f(x) = L$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ and $x \in A$, it follows that $|f(x) - L| < \epsilon$.

Theorem 4.2.3 (Sequential Criterion for Functional Limits)
Given a function $f: A \rightarrow \mathbb{R}$ and a limit point c of A , the following two statements are equivalent:
(i) $\lim_{x \rightarrow c} f(x) = L$.
(ii) For every sequence (x_n) in A such that $x_n \rightarrow c$ and $x_n \neq c$, it follows that $f(x_n) \rightarrow L$.

Algebraic Limit Theorem for Functional Limits
Algebraic Limit Theorem for Functional Limits:
 $\lim_{x \rightarrow c} (a_n x + b_n) = a \lim_{x \rightarrow c} x + b$
 $\lim_{x \rightarrow c} (a_n x^2 + b_n x + c_n) = a \lim_{x \rightarrow c} x^2 + b \lim_{x \rightarrow c} x + c$

4.3 Continuous Functions

Definition 4.3.1 (Continuity)
A function $f: A \rightarrow \mathbb{R}$ is continuous at a point $c \in A$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ and $x \in A$, it follows that $|f(x) - f(c)| < \epsilon$.

Corollary 4.3.3 (Criterion for Discontinuity)
Let $f: A \rightarrow \mathbb{R}$ and let $c \in A$ be a limit point of A . If there exists a sequence (x_n) in A where $x_n \rightarrow c$ but $f(x_n)$ does not converge to $f(c)$, we may conclude that f is not continuous at c .

Theorem 4.3.2 (Characterizations of Continuity)
Let $f: A \rightarrow \mathbb{R}$ and let $c \in A$. The function f is continuous at c if and only if any one of the following three conditions is met:
1. f is continuous at c .
2. f is continuous at c .
3. f is continuous at c .

4.4 Continuous Functions on Compact Sets

Theorem 4.4.1 (Preservation of Compact Sets)
Theorem 4.4.1 (Preservation of Compact Sets). Let $f: A \rightarrow \mathbb{R}$ be continuous on a compact set $K \subseteq A$. If $c \in A$ is compact, then $f(K)$ is compact as well.

Theorem 4.4.2 (Extreme Value Theorem)
Theorem 4.4.2 (Extreme Value Theorem). If $f: K \rightarrow \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then f attains a maximum and minimum value in other words, there exist $x_1, x_2 \in K$ such that $f(x_1) = \max_{x \in K} f(x)$ and $f(x_2) = \min_{x \in K} f(x)$.

4.5 The Intermediate Value Theorem

Intermediate Value Theorem
Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ (or $f(b) < L < f(a)$), then there exists a point $c \in (a, b)$ where $f(c) = L$.

intermediate value property
A function has the intermediate value property on an interval $[a, b]$ if for $c \in (a, b)$ and all y between $f(a)$ and $f(b)$, it is always possible to find a point $x \in (a, b)$ where $f(x) = y$.

Preservation of Connected Sets
Let $f: K \subseteq \mathbb{R}$ be continuous. If $K \subseteq \mathbb{R}$ is connected, then $f(K)$ is connected as well.

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Limit Superior
Limit Superior
 $\limsup_{n \rightarrow \infty} a_n = \inf_{k \in \mathbb{N}} \sup_{n \geq k} a_n$
The sequence $(1/n)$ converges if and only if $p > 1$.

Theorem 2.4.2 (Monotone Convergence Theorem)
If a sequence is monotone and bounded, then it converges.

Cauchy Condensation Test
Suppose (a_n) is a decreasing and positive sequence of real numbers. Then, the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

2.7 Properties of Infinite Series

Geometric Series
 $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $|r| < 1$.

Algebraic Limit Theorems for Series
Algebraic Limit Theorems for Series:
 $\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$
 $\lim_{n \rightarrow \infty} (c a_n) = c \lim_{n \rightarrow \infty} a_n$

Theorem 2.7.2 (Cauchy Criterion for Series)
The series $\sum_{n=1}^{\infty} a_n$ converges if and only if (s_n) is a Cauchy sequence. That is, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$, it follows that $|s_m - s_n| < \epsilon$.

Theorem 2.7.4 (Comparison Test)
Theorem 2.7.4 (Comparison Test). Assume (a_n) and (b_n) are sequences satisfying $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$.
1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Theorem 2.7.3
Theorem 2.7.3. If the series $\sum_{n=1}^{\infty} a_n$ converges, then (a_n) is a Cauchy sequence.

Theorem 2.7.6 (Absolute Convergence Test)
Theorem 2.7.6 (Absolute Convergence Test). If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.

converges absolutely
If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Abel's Test
Abel's Test for convergence states that if the series $\sum_{n=1}^{\infty} b_n$ converges and (a_n) is a sequence satisfying $|a_n| \leq M$ for all $n \in \mathbb{N}$, then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

Ratio Test
Given a series $\sum_{n=1}^{\infty} a_n$, let $r_n = |a_{n+1}/a_n|$. The Ratio Test states that if $\lim_{n \rightarrow \infty} r_n = L$, then the series converges absolutely if $L < 1$, diverges if $L > 1$, and the test is inconclusive if $L = 1$.

Alternating Series Test
(Alternating Series Test) Let (a_n) be a sequence satisfying (i) $a_n > 0$, (ii) $a_n \rightarrow 0$, and (iii) (a_n) is decreasing. Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Rearrangement
Let (a_n) be a series. A series $\sum_{n=1}^{\infty} a_n$ is said to be **rearranged** if the terms are rearranged in a different order. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges to the same sum for any permutation σ of \mathbb{N} .

Dini's Test
Dini's Test for convergence states that if the partial sums (s_n) are bounded and (a_n) is a sequence satisfying (i) $a_n > 0$, (ii) $a_n \rightarrow 0$, and (iii) (a_n) is decreasing, then $\sum_{n=1}^{\infty} a_n$ converges.

2.8 Double Summations and Products of Infinite Series

Double Summations
Double Summations and Products of Infinite Series:
 $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{nk} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{nk}$ if $\sum_{n,k} |a_{nk}| < \infty$.

5 The Derivative

5.2 Derivatives and the Intermediate Value Property

Darboux's Theorem
If f is differentiable on an interval $[a, b]$, then f' has the Intermediate Value Property on $[a, b]$.

Algebraic Differentiability Theorem
Let $f: A \rightarrow \mathbb{R}$ and $g: B \rightarrow \mathbb{R}$ be functions defined on an interval A , and assume both are differentiable at some point $c \in A$.
 $(f+g)'(c) = f'(c) + g'(c)$
 $(cf)'(c) = c f'(c)$
 $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$
 $(f/g)'(c) = \frac{f(c)g'(c) - f'(c)g(c)}{g(c)^2}$

Theorem 5.2.3
If $f: A \rightarrow \mathbb{R}$ is differentiable at a point $c \in A$, then f is continuous at c as well.

Interior Extremum Theorem
Let f be differentiable on an open interval (a, b) . If f attains a maximum value at some point $c \in (a, b)$, then $f'(c) = 0$. Similarly, if f attains a minimum value at some point $c \in (a, b)$, then $f'(c) = 0$.

5.3 The Mean Value Theorem

(Mean Value Theorem)
If $f: [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) , then there exists a point $c \in (a, b)$ where $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Rolle's Theorem
Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ where $f'(c) = 0$.

5.2 Derivatives and the Intermediate Value Property

Darboux's Theorem
If f is differentiable on an interval $[a, b]$, then f' has the Intermediate Value Property on $[a, b]$.

Theorem 5.2.5 chain rule
Let $f: A \rightarrow \mathbb{R}$ and $g: B \rightarrow \mathbb{R}$ be functions defined on an interval A , and assume both are differentiable at some point $c \in A$.
 $(f \circ g)'(c) = f'(g(c))g'(c)$

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6 Sequences and Series of Functions

6.2 Uniform Convergence of a Sequence of Functions

converges pointwise
For each $x \in \mathbb{R}$, let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbb{R}$. The sequence (f_n) converges pointwise on A to a function f if for all $x \in A$, the sequence of real numbers $(f_n(x))$ converges to $f(x)$.

Continuous Limit Theorem
Let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbb{R}$ that converges uniformly on A to a function f . If each f_n is continuous at $c \in A$, then f is continuous at c .

6.4 Series of Functions

term-by-term Continuity Theorem
Term-by-term Continuity Theorem. Let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbb{R}$ that converges uniformly on A to a function f . Then, f is continuous on A .

term-by-term Differentiability Theorem
(Term-by-term Differentiability Theorem) Let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbb{R}$ that converges uniformly on A to a function f . If each f_n is differentiable at a point $c \in A$, then f is differentiable at c and $f'(c) = \lim_{n \rightarrow \infty} f'_n(c)$.

Weierstrass M-Test
(Weierstrass M-Test) For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subseteq \mathbb{R}$, and let $M_n > 0$ be a real number satisfying

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Theorem 5.2.5 chain rule
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