Characterizing Alternating Groups by Their Codegree Sets

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Definition 1: Complex Representation

A complex representation of a group $G$ is a homomorphism $\rho : G \rightarrow \text{GL}(n, \mathbb{C})$ from $G$ to $n \times n$ invertible matrices over $\mathbb{C}$. A representation is irreducible if it has no proper, nontrivial subrepresentation.
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Definition 2: Character

The character of a complex representation $\rho$ is the map $\chi_\rho : g \mapsto \text{Tr}(\rho(g))$ where $\text{Tr}(\rho(g))$ is the trace of the matrix $\rho(g)$. We let $\text{Irr}(G)$ denote the set of irreducible characters of $G$. 

Definition 3: Codegree

The codegree of a character $\chi$ of a group $G$ is defined as $\text{cod}(\chi) := \frac{|G|}{|\ker(\chi)|}$ where $\ker(\chi) := \ker(\rho)$.
Preliminary Definitions

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**Conjecture (Kourovka Notebook Problem 20.79)**

If $H$ is a finite nonabelian simple group and $G$ is a finite group with $\text{cod}(G) = \text{cod}(H)$, then $G \cong H$. 

Recently, various authors have proved this conjecture for various simple groups such as $\text{PSL}(2, q)$, $2^{2f+1}B_2$, $\text{PSL}(3, 4)$, and all the sporadic groups. In this project, we verify this conjecture for any alternating group $A_n$ for $n \geq 5$.
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Suppose for contradiction that $G$ is a minimal counterexample to the conjecture. That is, $\text{cod}(G) = \text{cod}(A_n)$, but $G \not\cong A_n$. 

Let $N$ be a maximal normal subgroup of $G$. Then $G/N$ is simple, and $\text{cod}(G/N) \subseteq \text{cod}(G) = \text{cod}(A_n)$. 

We first show that $G/N \cong A_n$. Then, we show that $N = 1$, so $G \cong G/N \cong A_n$.
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Proof Outline

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Then, we show that $N = 1$, so $G \cong G/N \cong A_n$. 
We show that if $G/N \not\cong A_n$, we get a contradiction by proving $\text{cod}(G/N) \not\subseteq \text{cod}(A_n)$ for all other nonabelian simple groups $G/N$. 
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**Lemma 5: Dividing Orders**

Let $H$ be a finite group and $K$ be a finite simple group with $\text{cod}(K) \subseteq \text{cod}(H)$, then $|K|$ divides $|H|$.
We show that if $G/N \not\cong A_n$, we get a contradiction by proving $\text{cod}(G/N) \not\subseteq \text{cod}(A_n)$ for all other nonabelian simple groups $G/N$.

**Lemma 5: Dividing Orders**

Let $H$ be a finite group and $K$ be a finite simple group with $\text{cod}(K) \subseteq \text{cod}(H)$, then $|K|$ divides $|H|$.

**Lemma 6: Order Bound**

Let $H$ be a finite simple group and $K$ be a finite group with $\text{cod}(K) \subseteq \text{cod}(H)$. Then, $|H| < |K| \cdot |\text{Irr}(K)|$.
Proving $G/N \cong A_n$

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**Proof.** $\forall \chi \in \text{Irr}(H), \chi(1)^2 < |H|$. $H$ is simple so $\ker(\chi) = 1$ and $\text{cod}(\chi) = \frac{|H|}{\chi(1)} > \sqrt{|H|}$. Then, $\text{cod}(K) \subseteq \text{cod}(H)$ implies that $\forall \psi \in \text{Irr}(K), \frac{|K|}{\psi(1)} > \text{cod}(\psi) > \sqrt{|H|}$. Thus, $\psi(1) < \frac{|K|}{\sqrt{|H|}}$. Summing the squares of irreducible character degrees gives $|K| < |\text{Irr}(K)| \frac{|K|^2}{|H|}$ and the inequality follows.
**Corollary 7: Order Restrictions**

If \( \text{cod}(G/N) \subseteq \text{cod}(A_n) \), then \( \frac{|A_n|}{|G/N|} \in \{1, 2, \ldots, |\text{Irr}(G/N)|\} \).
Corollary 7: Order Restrictions

If $\text{cod}(G/N) \subseteq \text{cod}(A_n)$, then $\frac{|A_n|}{|G/N|} \in \{1, 2, \ldots, |\text{Irr}(G/N)|\}$.

- For each sporadic (or Tits) group, $G/N$, we obtain an upper bound on $n$ by using $|A_n| < |G/N| \cdot |\text{Irr}(G/N)|$.
- Then we can computationally check which values of $n$ satisfy $|G/N|$ divides $|A_n|$.
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- The only possibility is \( A_n = A_{10} \) and \( G/N \cong J_2 \),
- In this case, we can check using the ATLAS that \( \text{cod}(J_2) \not\subseteq \text{cod}(A_{10}) \).
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Thus if \( \text{cod}(G/N) \subseteq \text{cod}(A_n) \), \( G/N \) cannot be a sporadic group or the Tits group.
Example 8: $\text{PSL}(m + 1, q)$

Let $G/N = \text{PSL}(m + 1, q)$, where $m$ is a positive integer and $q = p^k$ is a prime power. Then, $|G/N| = \frac{q^{m(m+1)/2}}{\gcd(m+1, q-1)} \prod_{i=1}^{m} (q^{i+1} - 1)$ and $|\text{Irr}(G/N)| \leq 2.5q^m$. 
Example 8: \( \text{PSL}(m + 1, q) \)

Let \( G/N = \text{PSL}(m + 1, q) \), where \( m \) is a positive integer and \( q = p^k \) is a prime power. Then, \( |G/N| = \frac{q^{m(m+1)/2}}{\gcd(m+1,q-1)} \prod_{i=1}^{m} (q^{i+1} - 1) \) and \( |\text{Irr}(G/N)| \leq 2.5q^m \).

**Proof.** We know \( q^{m(m+1)/2} \) divides \( |G/N| \) and that \( |A_n|_q \leq \frac{n}{k(p-1)} \). Since \( |G/N| \) divides \( |A_n| \), we have \( \frac{m(m+1)}{2} \leq \frac{n}{k(p-1)} \). Thus, \( n \geq \frac{m(m+1)k(p-1)}{2} \).

Using the Lemma which limits \( |A_n| \) and the above inequality, we have \( |A_{m(m+1)k(p-1)}| < |G/N| \cdot 2.5q^m \). As functions of \( m, p, \) or \( k \) only, the left-hand side grows faster asymptotically than the right so we find maximum values for each of these variables for which the inequality is satisfied. Namely, \( m \leq 6, p \leq 17, \) and \( k \leq 63 \). Now, we can analyze each of these possible combinations in turn in the same way we did for the sporadic groups.
We get the following list of possible exceptions which satisfy
\[ |\text{PSL}(m + 1, q)| \text{ divides } |A_n| \text{ and } |A_n| < |\text{PSL}(m + 1, q)| \cdot 2.5q^m: \]

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**Remark.** $\text{PSL}(2, 4) \cong \text{PSL}(2, 5) \cong A_5$, $\text{PSL}(2, 9) \cong A_6$, and $\text{PSL}(4, 2) \cong A_8$.

- In all other cases, when $\text{PSL}(m + 1, q) \ncong A_n$, we can check using the ATLAS that $\text{cod}(\text{PSL}(m + 1, q)) \ncong \text{cod}(A_n)$.

- We follow these same steps to show that for $G/N$ any simple group of Lie type, $\text{cod}(G/N) \ncong \text{cod}(A_n)$.
If $G/N \cong A_m \not\cong A_n$, then we know $m < n$, since $|A_m|$ divides $|A_n|$. 
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Let $a_x$ denote the the minimal nontrivial codegree of $A_x$. We will show that $a_m < a_n$, proving that if $\text{cod}(A_m) \subseteq \text{cod}(A_n)$, then $m = n$. 
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Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ be a partition of $n$ and $V_\lambda$ be the corresponding irreducible representation of $S_n$. 
Interpreting $\lambda$ as a Young diagram with $k$ rows of length $\lambda_1 \ldots \lambda_k$, we have that

$$\text{cod}(V_{\lambda}) = \frac{n!}{\text{dim}(V_{\lambda})} = \prod h_{\lambda}(i, j) := H_{\lambda},$$

where the product goes over all the cells of the Young diagram, and $h_{\lambda}(i, j)$ is the hook length of the cell $(i, j)$, i.e. the number of cells $(a, b)$ such that $a = i$ and $b \geq j$ or $a \geq i$ and $b = j$. 
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Let $U_\lambda$ be the restriction of $V_\lambda$ to $A_n$. If $\lambda$ is not self-conjugate, then $U_\lambda$ remains irreducible, and, for our purposes, we can prove that we only need to consider $\lambda$ which are not self-conjugate.
Thus \( a_n = H_\lambda/2 \) for some \( \lambda \). Then for \( m > 2 \), we may choose a non-self-conjugate partition of \( m \), \( \mu \), such that \( \mu \) is completely contained in \( \lambda \). Then

\[
a_m \leq \frac{H_\mu}{2} < \frac{H_\lambda}{2} = a_n
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However, $a_m \in \text{cod}(A_m) \subseteq \text{cod}(A_n)$ while $a_n$ is the smallest nontrivial element of $\text{cod}(A_n)$ and hence we get a contradiction. Thus if $G/N$ is an alternating group and $\text{cod}(G/N) \subseteq \text{cod}(A_n)$, then $G/N \cong A_n$. 
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Thus, we have shown in the previous slides that $\text{cod}(G/N) \not\subseteq \text{cod}(A_n)$ for any simple group $G/N$ except $G/N \cong A_n$. Thus $G/N \cong A_n$. 
Let $G$ be a minimal counterexample where $\text{cod}(G) = \text{cod}(A_n)$ but $G \not\cong A_n$. Let $N$ be a maximal normal subgroup of $G$. Then, $G/N \cong A_n$. We prove the following relatively simple properties:
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- $N$ is also a minimal normal subgroup
- $N$ is the unique minimal normal subgroup of $G$ which implies it is the only proper, nontrivial normal subgroup of $G$
- Every character $\chi \in \text{Irr}(G \mid N) := \text{Irr}(G) - \text{Irr}(G/N)$ is faithful
Proving \( N = 1 \)

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- \( N \) is an elementary abelian group
- \( C_G(N) = N \) (Schur multiplier of \( A_n \))
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- $N$ is an elementary abelian group
- $C_G(N) = N$ (Schur multiplier of $A_n$)
- $\frac{|I_G(\lambda)|}{\theta(1)} \in \text{cod}(G)$ and $|N|$ divides $|G/N|$
Summary

$|N| = p^m$ divides $|G/N| = |A_n|$. By the normalizer-centralizer theorem, $A_n \cong G/N = N_G(N)/C_G(N) \leq \text{Aut}(N)$ and $m > 1$. In general, $\text{Aut}(N) = \text{GL}(m, p)$, so $A_n \leq \text{GL}(m, p)$. 

For $n > 7$, the minimal faithful representation of $A_n$ over a finite field has degree at least $n - 2$. I.E. $m \geq n - 2$.

Thus, we deduce from these inequalities that $p = 2$ and $m = n - 2$. 

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- The $p$-part of $n!$ gives $|A_n|_p \leq \frac{n}{p-1}$ for any prime $p$. I.E. $m \leq \frac{n}{p-1}$
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Let \( \lambda \in \text{Irr}(N) \), \( \vartheta \in \text{Irr}(I_G(\lambda)|\lambda) \), and \( T := I_G(\lambda) \)
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\[ 1 < |G : T| < |N| = 2^{n-2} \text{ for } |G : T| \text{ is the number of all conjugates of } \lambda \]
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\[ \frac{|T|}{\vartheta(1)} \in \text{cod}(G) \] and \( |N| \) divides \( \frac{|T|}{\vartheta(1)} \)
\begin{itemize}
  \item $|N|_2 = |A_n|_2 = m - 2$
  \item Let $\lambda \in \text{Irr}(N)$, $\vartheta \in \text{Irr}(I_G(\lambda)|\lambda)$, and $T := I_G(\lambda)$
  \item $1 < |G : T| < |N| = 2^{n-2}$ for $|G : T|$ is the number of all conjugates of $\lambda$
  \item $\frac{|T|}{\vartheta(1)} \in \text{cod}(G)$ and $|N|$ divides $\frac{|T|}{\vartheta(1)}$
  \item $\left| \frac{|T/N|}{\vartheta(1)} \right|_2 = 0$ so the 2-parts of $|T/N|$ and $\vartheta(1)$ are equal
\end{itemize}
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$\left| \frac{|T/N|}{\vartheta(1)} \right|_2 = 0$ so the 2-parts of $|T/N|$ and $\vartheta(1)$ are equal

$|T/N|$ is a sum of squares of the form $\vartheta(1)$ so the 2-parts of each must be 1
\[ |N|_2 = |A_n|_2 = m - 2 \]

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\[ \frac{|T|}{\vartheta(1)} \in \text{cod}(G) \text{ and } |N| \text{ divides } \frac{|T|}{\vartheta(1)} \]

\[ \left( \frac{|T/N|}{\vartheta(1)} \right)_2 = 0 \] so the 2-parts of \( |T/N| \) and \( \vartheta(1) \) are equal

\[ |T/N| \] is a sum of squares of the form \( \vartheta(1) \) so the 2-parts of each must be 1

Thus, since \( |G/N|_2 \geq |N|_2 = n - 2 \), we have
\[ |G : T|_2 = |G/N : T/N|_2 \geq n - 2 \]
and so \( |G : T| \geq 2^{n-2} = |N| \), which is a contradiction.


[7] N. N. Hung, Group pseudo-algebras of finite simple groups. *In progress*
