Characterizing Alternating Groups by Their Codegree Sets

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Preliminary Definitions

Definition 1: Complex Representation

A complex *representation* of a group G is a homomorphism $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ from G to $n \times n$ invertible matrices over $\mathbb{C}$. A representation is *irreducible* if it has no proper, nontrivial subrepresentation.

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Definition 2: Character

The *character* of a complex representation ρ is the map $\chi_\rho : g \mapsto \mathrm{Tr}(\rho(g))$ where $\mathrm{Tr}(\rho(g))$ is the trace of the matrix $\rho(g)$. We let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G.

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Definition 3: Codegree

The *codegree* of a character χ of a group G is defined as $\mathrm{cod}(\chi) := \frac{|G:\ker(\chi)|}{\chi(1)}$, where $\ker(\chi) := \ker(\rho)$.

Definition 4: Codegree Set

The *codegree set* of a group G is defined as $\mathrm{cod}(G) := \{\mathrm{cod}(\chi) | \chi \in \mathrm{Irr}(G)\}$.

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Conjecture (Kourovka Notebook Problem 20.79)

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Recently, various authors have proved this conjecture for various simple groups such as $\mathrm{PSL}(2, q), {}^2B_2(2^{2f+1}), \mathrm{PSL}(3, 4)$, and all the sporadic groups. In this project, we verify this conjecture for H any alternating group A_n for $n \geq 5$.

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- Suppose for contradiction that G is a minimal counterexample to the conjecture. That is, $\mathrm{cod}(G) = \mathrm{cod}(\mathrm{A}_n)$, but $G \not\cong \mathrm{A}_n$.

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- Let N be a maximal normal subgroup of G. Then G/N is simple, and $\mathrm{cod}(G/N) \subseteq \mathrm{cod}(G) = \mathrm{cod}(\mathrm{A}_n)$.
- We first show that $G/N \cong \mathrm{A}_n$.
- Then, we show that $N = 1$, so $G \cong G/N \cong \mathrm{A}_n$.

Proving $G/N \cong \mathrm{A}_n$

We show that if $G/N \ncong \mathrm{A}_n$, we get a contradiction by proving $\mathrm{cod}(G/N) \not\subseteq \mathrm{cod}(\mathrm{A}_n)$ for all other nonabelian simple groups G/N.

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Lemma 5: Dividing Orders

Let H be a finite group and K be a finite simple group with $\mathrm{cod}(K) \subseteq \mathrm{cod}(H)$, then $|K|$ divides $|H|$.

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Let H be a finite group and K be a finite simple group with $\mathrm{cod}(K) \subseteq \mathrm{cod}(H)$, then $|K|$ divides $|H|$.

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Let H be a finite simple group and K be a finite group with $\mathrm{cod}(K) \subseteq \mathrm{cod}(H)$. Then, $|H| < |K| \cdot |\mathrm{Irr}(K)|$

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Proof. $\forall \chi \in \mathrm{Irr}(H), \chi(1)^2 < |H|$. H is simple so $\ker(\chi) = 1$ and $\mathrm{cod}(\chi) = \frac{|H|}{\chi(1)} > \sqrt{|H|}$. Then, $\mathrm{cod}(K) \subseteq \mathrm{cod}(H)$ implies that $\forall \psi \in \mathrm{Irr}(K), \frac{|K|}{\psi(1)} > \mathrm{cod}(\psi) > \sqrt{|H|}$. Thus, $\psi(1) < \frac{|K|}{\sqrt{|H|}}$. Summing the squares of irreducible character degrees gives $|K| < |\mathrm{Irr}(K)| \frac{|K|^2}{|H|}$ and the inequality follows.

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Corollary 7: Order Restrictions

If $\mathrm{cod}(G/N) \subseteq \mathrm{cod}(\mathrm{A}_n)$, then $\frac{|\mathrm{A}_n|}{|G/N|} \in \{1, 2, \ldots, |\mathrm{Irr}(G/N)|\}$.

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- For each sporadic (or Tits) group, G/N, we obtain an upper bound on n by using $|\mathrm{A}_n| < |G/N| \cdot |\mathrm{Irr}(G/N)|$.
- Then we can computationally check which values of n satisfy $|G/N|$ divides $|\mathrm{A}_n|$.

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- Then we can computationally check which values of n satisfy $|G/N|$ divides $|\mathrm{A}_n|$.
- The only possibility is $\mathrm{A}_n = \mathrm{A}_{10}$ and $G/N \cong \mathrm{J}_2$,
- In this case, we can check using the ATLAS that $\mathrm{cod}(J_2) \not\subseteq \mathrm{cod}(\mathrm{A}_{10})$.

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- In this case, we can check using the ATLAS that $\operatorname{cod}(J_2) \not\subseteq \operatorname{cod}(\mathrm{A}_{10})$.

Thus if $\operatorname{cod}(G/N) \subseteq \operatorname{cod}(\mathrm{A}_n)$, G/N cannot be a sporadic group or the Tits group.

Example 8: $\mathrm{PSL}(m+1, q)$

Let $G/N = \mathrm{PSL}(m+1, q)$, where m is a positive integer and $q = p^k$ is a prime power. Then, $|G/N| = \frac{q^{m(m+1)/2}}{\gcd(m+1, q-1)} \prod_{i=1}^{m}(q^{i+1} - 1)$ and $|\mathrm{Irr}(G/N)| \leq 2.5q^m$.

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Proof. We know $q^{m(m+1)/2}$ divides $|G/N|$ and that $|\mathrm{A}_n|_q \leq \frac{n}{k(p-1)}$. Since $|G/N|$ divides $|\mathrm{A}_n|$, we have $\frac{m(m+1)}{2} \leq \frac{n}{k(p-1)}$. Thus, $n \geq \frac{m(m+1)k(p-1)}{2}$.

Using the Lemma which limits $|\mathrm{A}_n|$ and the above inequality, we have $|\mathrm{A}_{\frac{m(m+1)k(p-1)}{2}}| < |G/N| \cdot 2.5q^m$. As functions of m, p, or k only, the left-hand side grows faster asymptotically than the right so we find maximum values for each of these variables for which the inequality is satisfied. Namely, $m \leq 6, p \leq 17$, and $k \leq 63$. Now, we can anaylze each of these possible combinations in turn in the same way we did for the sporadic groups.

G/N is a Simple Group of Lie Type, Part 2

Possible Exceptions

We get the following list of possible exceptions which satisfy $|\mathrm{PSL}(m+1,q)|$ divides $|\mathrm{A}_n|$ and $|\mathrm{A}_n| < |\mathrm{PSL}(m+1,q)| \cdot 2.5q^m$:

m	q	n
1	4	5
1	4	6
1	8	7
1	9	6
1	9	7
1	5	5
1	5	6
1	7	7
2	4	8
2	4	9
3	2	8
3	2	9

Remark. $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5) \cong \mathrm{A}_5$, $\mathrm{PSL}(2,9) \cong \mathrm{A}_6$, and $\mathrm{PSL}(4,2) \cong \mathrm{A}_8$.

- In all other cases, when $\mathrm{PSL}(m+1,q) \not\cong \mathrm{A}_n$, we can check using the ATLAS that $\mathrm{cod}(\mathrm{PSL}(m+1,q)) \not\subseteq \mathrm{cod}(\mathrm{A}_n)$.
- We follow these same steps to show that for G/N any simple group of Lie type, $\mathrm{cod}(G/N) \not\subseteq \mathrm{cod}(\mathrm{A}_n)$.

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Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ be a partition of n and V_λ be the corresponding irreducible representation of S_n.

G/N is an Alternating Group

Interpreting λ as a Young diagram with k rows of length $\lambda_1 \ldots \lambda_k$, we have that

$$\mathrm{cod}(V_\lambda) = \frac{n!}{\dim(V_\lambda)} = \prod h_\lambda(i,j) := H_\lambda,$$

where the product goes over all the cells of the Young diagram, and $h_\lambda(i,j)$ is the hook length of the cell (i,j), i.e. the number of cells (a,b) such that $a = i$ and $b \geq j$ or $a \geq i$ and $b = j$.

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Let U_λ be the restriction of V_λ to A_n. If λ is not self-conjugate, then U_λ remains irreducible, and, for our purposes, we can prove that we only need to consider λ which are not self-conjugate.

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Thus $a_n = H_\lambda/2$ for some λ. Then for $m > 2$, we may choose a non-self-conjugate partition of m, μ, such that μ is completely contained λ. Then

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However, $a_m \in \mathrm{cod}(\mathrm{A}_m) \subseteq \mathrm{cod}(\mathrm{A}_n)$ while a_n is the smallest nontrivial element of $\mathrm{cod}(\mathrm{A}_n)$ and hence we get a contradiction. Thus if G/N is an alternating group and $\mathrm{cod}(G/N) \subseteq \mathrm{cod}(\mathrm{A}_n)$, then $G/N \cong \mathrm{A}_n$.

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Thus, we have shown in the previous slides that $\mathrm{cod}(G/N) \not\subseteq \mathrm{cod}(\mathrm{A}_n)$ for any simple group G/N except $G/N \cong \mathrm{A}_n$. Thus $G/N \cong \mathrm{A}_n$.

Proving $N = 1$

Let G be a minimal counterexample where $\mathrm{cod}(G) = \mathrm{cod}(\mathrm{A}_n)$ but $G \not\cong \mathrm{A}_n$. Let N be a maximal normal subgroup of G. Then, $G/N \cong \mathrm{A}_n$. We prove the following relatively simple properties:

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- N is also a minimal normal subgroup
- N is the unique minimal normal subgroup of G which implies it is the only proper, nontrivial normal subgroup of G
- Every character $\chi \in \mathrm{Irr}(G \mid N) := \mathrm{Irr}(G) - \mathrm{Irr}(G/N)$ is faithful

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- $\mathbf{C}_G(N) = N$ (Schur multiplier of A_n)

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- N is an elementary abelian group
- $\mathbf{C}_G(N) = N$ (Schur multiplier of A_n)
- $\frac{|I_G(\lambda)|}{\theta(1)} \in \mathrm{cod}(G)$ and $|N|$ divides $|G/N|$

Proving $N = 1$, Part 2

Summary

$|N| = p^m$ divides $|G/N| = |\mathrm{A}_n|$. By the normalizer-centralizer theorem, $\mathrm{A}_n \cong G/N = \mathbf{N}_G(N)/\mathbf{C}_G(N) \leq \mathrm{Aut}(N)$ and $m > 1$. In general, $\mathrm{Aut}(N) = \mathrm{GL}(m, p)$, so $\mathrm{A}_n \lesssim \mathrm{GL}(m, p)$.

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- Thus, we deduce from these inequalities that $p = 2$ and $m = n-2$

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- $\left|\frac{|T/N|}{\vartheta(1)}\right|_2 = 0$ so the 2-parts of $|T/N|$ and $\vartheta(1)$ are equal
- $|T/N|$ is a sum of squares of the form $\vartheta(1)$ so the 2-parts of each must be 1
- Thus, since $|G/N|_2 \geq |N|_2 = n - 2$, we have $|G : T|_2 = |G/N : T/N|_2 \geq n - 2$ and so $|G : T| \geq 2^{n-2} = |N|$, which is a contradiction.

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